SOLVING MAX-MIN PROBS (4.6)

20-41

14 p 264

cf. pg 240 198-199

1. Translate problem into an equation, after drawing a picture when helpful.

14. CONSTRAINTS

2. Find $\frac{dy}{dx}$ and solve $\frac{dy}{dx}=0$ for x to find "critical points" at $x_1, x_2, \ldots$.

EXA

3. Find $\frac{d^2y}{dx^2}$ and test the critical points to see whether $\left.\frac{d^2y}{dx^2}\right|_{x_1} > 0$ or < 0.

   If >0, you have a min
   If <0, you have a max.
   (If $=0$, use another test)

4. Always check function values at endpoints (if $f(x)$ is defined for a limited range!) → EX 1

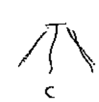
PROBLEM CASES.

NEXT DAY

1. Check points where deriv fails to exist.
   (eg. if $f(x)=|x|$, no deriv at $x=0$, but that is min.!)

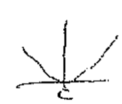
2. If $\frac{d^2y}{dx^2}=0$, look at first deriv $\frac{dy}{dx}$. If c is a critical point.

   (a) if $\left.\frac{dy}{dx}\right|_{x<c} > 0$ and $\left.\frac{dy}{dx}\right|_{x>c} < 0$.



   then c is a max.

   (b) if $\left.\frac{dy}{dx}\right|_{x<c} < 0$ and $\left.\frac{dy}{dx}\right|_{x>c} > 0$



   Then c is a min.

EXAMPLES

EX 1

Find 2 pos #'s, sum = 20, product is as large as possible.

Let x be 1 number and u the other.

$\therefore \; x + u = 20$

or $\quad u = 20 - x$

Let $y = f(x) = x \cdot u = x(20-x) = 20x - x^2$

We want to maximize y.

$$\frac{dy}{dx} = 20 - 2x = 2(10 - x)$$

$\therefore$ critical pt is $x = 10$

$\left(\frac{d^2y}{dx^2} = -2 \quad \text{(always)} \quad \therefore \; y \text{ is maximized at } x = 10. \right)$

What about end points?

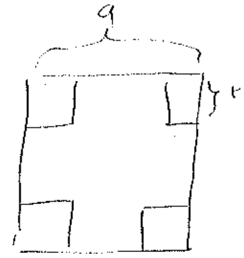
$x = 0, \; x = 20 \quad \Rightarrow f(x) = y = 0$

Ex 4

p 264

12

square sheet a inches on a side.
used to make open-top box w max volume.
How much do we cut out of the corners.



$$V = l \cdot w \cdot h$$
$$= (a-2x)(a-2x)x$$
$$= x(a-2x)^2$$

$$\frac{dV}{dx} = x \cdot 2 \cdot (a-2x)(-2) + (a-2x)^2$$
$$= -4x(a-2x) + (a-2x)^2$$
$$= (a-2x)(a-2x-4x)$$
$$= (a-2x)(a-6x) = a^2 - 8ax + 12x^2$$

Set equal to 0

$$\Rightarrow (a-2x)(a-6x) = 0$$
$$\Rightarrow a-2x=0 \quad \text{or} \quad a-6x=0$$
$$a = 2x \quad \text{or} \quad a = 6x$$
$$x = \frac{a}{2} \overset{12}{=} 6 \quad \text{or} \quad x = \frac{a}{6} \overset{12}{=} 2$$

$x = \frac{a}{2}$ makes no sense
since if x were that large, there would be no box!

According to Thm 1, the max/min point must be in the interior of the interval $[0, \frac{a}{2}]$, not one of the endpoints.

Try 2nd deriv test.

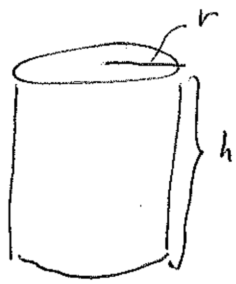
$$\frac{d^2V}{dx^2} = -8a + 24a = 24(x - \frac{a}{3})$$

for $x = \frac{a}{2}$ $\quad \frac{d^2V}{dx^2} > 0$ $\therefore$ concave up $\therefore$ min.

$x = \frac{a}{6}$ $\quad \frac{d^2V}{dx^2} < 0$ $\therefore$ concave down $\therefore$ max.

$\therefore$ we cut out squares of size $\frac{a}{6} \times \frac{a}{6}$ to max vol of box.

EX 2
p 215



Find dimensions for a right-circular cylindrical can with volume of 1 liter = 1000 cm²
that will use the least material.

Assume we can ignore any waste in material.

$$V = (\text{Area circle}) \cdot h = \pi r^2 h \Rightarrow 1000 = \pi r^2 h \Rightarrow h = \frac{1000}{\pi r^2}$$

Amount of material = Surface area of side + top + bottom

$$\therefore\ A = \underbrace{2\pi r \cdot h}_{\text{side}} + \underbrace{2\pi r^2}_{\text{top + bottom}}$$

$$= 2\cancel{\pi}\cancel{r} \cdot \frac{1000}{\cancel{\pi} r^{\cancel{2}}} + 2\pi r^2$$

$$= 2000 r^{-1} + 2\pi r^2$$

$$\frac{dA}{dr} = -2000 r^{-2} + 4\pi r = 4\pi r - \frac{2000}{r^2}$$

$$\Rightarrow 0 = \frac{4\pi r^3 - 2000}{r^2} \Rightarrow 4\pi r^3 - 2000 = 0$$

$$\Rightarrow 4\pi r^3 = 2000 \Rightarrow r^3 = \frac{500}{\pi} \Rightarrow r = \sqrt[3]{\frac{500}{\pi}}$$

$$h = \frac{1000}{\pi r^2} = \frac{2 \cdot 500}{\pi} \cdot \frac{\sqrt[3]{\pi^2}}{\sqrt[3]{500^2}} = 2 \cdot \frac{\sqrt[3]{500^3}}{\sqrt[3]{\pi^3}} \cdot \frac{\sqrt[3]{\pi^2}}{\sqrt[3]{500^2}}$$

$$= 2\frac{\sqrt[3]{500}}{\sqrt[3]{\pi}} = 2r$$

Is this max or min?

$$\frac{d^2A}{dr^2} = 4000 r^3 + 4\pi. \quad \text{Since } r > 0,\ \frac{d^2A}{dr^2} > 0$$

$\therefore$ the results give us a min.

Variations: ① what about open top?
② what if one considers waste?

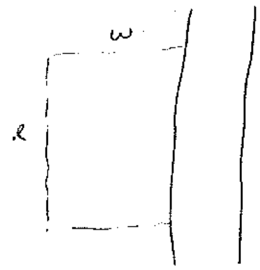
87 − 17
prob 1

88 − 21A

MAX-MIN EXAMPLES

p. 270
\# 7

- Open field next to straight river
- given length of fence $= f = 800$ m
- want to enclose largest area (rectangle) using fence for 3 sides.



Let $w =$ width

$\therefore\ l = \text{length} = f - 2w$

$$A = \text{area} = l \cdot w = w(f-2w) = wf - 2w^2$$

$$\frac{dA}{dw} = f - 4w = 0$$

$$\Rightarrow f = 4w \;=\; w = \frac{f}{4}$$

$$\frac{d^2A}{dw^2} = -4 < 0 \qquad \Rightarrow \text{concave down} \therefore \text{max.}$$

$$\text{width} = \underline{\underline{f/4}} = \frac{800}{4} = 200$$

$$\text{length} = f - 2\left(\frac{f}{4}\right) = f - \frac{f}{2} = \underline{\underline{\frac{f}{2}}} = \frac{800}{2} = 400$$

$$\text{Area} = 200 \times 400 = 80{,}000 \text{ m}^2$$