PRELIM

$\Rightarrow$ LIFE IS NEVER SIMPLE

We know certain derivative rules, e.g.

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(\sin x) = \cos x$$

BUT rarely do we encounter such simple expressions in applied problems.

We need to be able to handle derivatives of

$$(x^2+3)^2 \quad \text{or} \quad e^{x^2} \quad \text{or} \quad \sin x^2$$

This is the importance of the "CHAIN RULE" — maybe the most important rule in Calc I!

CHAIN RULE FOR COMPOSITE FUNCTIONS (3.6)

p 162

__Thm 2__ If $f(u)$ is differentiable at $u = g(x)$ and $g(x)$ is differentiable at x, then $(f \circ g)(x) = f(g(x))$ is differentiable at x and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

OR

if $y = f(u)$ and $u = g(x)$ then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

NOTE: derivatives are taken wrt different variables.

__Proof__ 1 p 164 → problematic

2 (3.11) p 209

B EXAMPLES

7th ed PS #40 15-10, 33 — NOT in 8th ed.

$y = x^{2/3} \quad x = t^2 + 1$

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \qquad \text{Now } \frac{dy}{dx} = \frac{2}{3}x^{-1/3} \text{ and } \frac{dx}{dt} = 2t$$

$$\therefore \frac{dy}{dt} = \left(\frac{2}{3}x^{-1/3}\right)(2t)$$

$$= \left(\frac{2}{3}\right)(t^2+1)^{-1/3}(2t) = \frac{4t}{3}(t^2+1)^{-1/3} \quad //$$

$$= \frac{4t}{3(t^2+1)^{1/3}} \quad //$$

Eg – can re-write longer eq to make use of chain rule.

$$y = (2x^2-1)^5 + 5(2x^2-1)^3 + (2x^2-1)$$

Let $u = 2x^2 - 1$

Thus $y = u^5 + 5u^3 + u$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \qquad \frac{dy}{du} = 5u^4 + 15u^2 + 1$$

$$\frac{du}{dx} = 4x$$

$$\therefore \frac{dy}{dx} = \left[5(2x^2-1)^4 + 15(2x^2-1)^2 + 1\right]4x \quad //$$

Eg

can use chain rule implicitly

$$y = \frac{(5x^3+5x^2)^5}{(4x^3-2)^3}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(4x^3-2)^3\, 5(5x^3+5x^2)^4(15x^2+10x) - (5x^3+5x^2)^5 \cdot 3(4x^3-2)^2(12x^2)}{(4x^3-2)^6}$$

RETHINKING OLD RULES

① Power rule: $y = x^n \Rightarrow \frac{dy}{dx} = nx^{n-1}$

with chain rule: $y = u^n$ (when $u = f(x)$)

$$\Rightarrow \frac{dy}{dx} = nu^{n-1} \cdot \frac{du}{dx}$$

② Sine rule: $y = \sin x \Rightarrow \frac{dy}{dx} = \cos x$

with chain rule: $y = \sin u \Rightarrow \frac{dy}{dx} = (\cos u) \cdot \frac{du}{dx}$

E.G. NOTE: always work from outside to inside. I.e. Last function evaluated is first to be differentiated

① $y = 5(x^3 + x^2 - 1)^3$

$$\frac{dy}{dx} = 5 \cdot 3(x^3 + x^2 - 1)^2 (3x^2 + 2x)$$

② $y = \sin x^2 \quad (\neq \sin^2 x)$

$$\frac{dy}{dx} = (\cos(x^2)) \cdot 2x = 2x \cos x^2$$

③ $y = e^{x^2}$

$$\frac{dy}{dx} = e^{x^2} \cdot 2x$$

④ $y = \sin^2 x = (\sin x)^2$

$$\frac{dy}{dx} = 2 \sin x \cos x$$

(NOT THE SAME AS #2!)

ADDITIONAL COMMENT ON CHAIN RULE

$$y = \sin(\cos(3x^2))$$

can be seen as 3 composite functions.

$$y = \sin u$$
$$u = \cos w$$
$$w = 3x^2$$

Thus $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dw} \cdot \frac{dw}{dx}$ by the chain rule.

We can also do the derivative in one step remembering we need to keep taking the derivative of the "guts" if it is "NOT A PLAIN OLD ORDINARY X - ALL BY ITSELF"

Thus.

$$\frac{dy}{dx} = \cos(\underbrace{\cos(3x^2)}_{\text{original angle (function)}})(-\sin(\underbrace{3x^2}_{\text{original angle (function)}}))\,6x$$

$$y = \sin^2(3\cos(3x^2))$$

$$\frac{dy}{dx} = 2\sin(3\cos(3x^2))\cos(3\cos(3x^2))(-3\sin(3x^2))\,6x$$

MORE EXAMPLES

EX 8 p 165 Show that the slope of every tangent line to $y = \frac{1}{(1-2x)^3}$ is positive.

Sol.

$$\frac{dy}{dx} = \frac{d}{dx}\left[(1-2x)^{-3}\right]$$

$$= -3(1-2x)^{-4}(-2)$$

$$= \frac{6}{(1-2x)^4}$$

num is pos = 6
denom is pos – even power
pos prob is $x=\frac{1}{2}$, but y is not defined there, so no need
to worry about deriv.

p 167 #96 Find tangent to $y = \sqrt{x^2-x+7}$ at $x=2$

$$\frac{dy}{dx} = \frac{d}{dx}(x^2-x+7)^{1/2} = \frac{1}{2}(x^2-x+7)^{-1/2}(2x-1) = \frac{2x-1}{2\sqrt{x^2-x+7}}$$

$$\left.\frac{dy}{dx}\right|_{x=2} = \frac{2\cdot 2-1}{2\sqrt{4-2+7}} = \frac{4-1}{2\sqrt{2+7}} = \frac{3}{2\sqrt{9}} = \frac{\cancel{3}}{2\cdot\cancel{3}} = \frac{1}{2}$$

$$y\Big|_{x=2} = \sqrt{4-2+7} = \sqrt{9} = 3$$

$\therefore$ tangent line is $\frac{y-3}{x-2} = \frac{1}{2}$

$$\Rightarrow \quad y-3 = \frac{1}{2}x - \frac{1}{2}\cdot 2$$

$$y = \boxed{\frac{1}{2}x - 1 + 3 = \frac{x}{2} + 2}$$