SIAM J. Appl. Math.
Vol. 70, No. 6, pp. 1977–1998

BALANCING SMALL TRANSACTION COSTS WITH LOSS OF OPTIMAL ALLOCATION IN DYNAMIC STOCK TRADING STRATEGIES*

JONATHAN GOODMAN† AND DANIEL N. OSTROV‡

Abstract. We discuss optimal trading strategies for general utility functions in portfolios of cash and stocks subject to small proportional transaction costs. We present a new interpretation of scalings found by Soner, Shreve, and others. To leading order in the small transaction cost parameter, the free boundary problem for the expected utility's value function is shown to be dual, in the sense of Lagrange multipliers for optimal design problems, to a free boundary problem minimizing a cost function. This cost function is the sum of a boundary integral corresponding to the rate of trading and an interior integral corresponding to opportunity loss that results from suboptimal portfolio allocation. Using the dual problem's formulation, we show that the quasi-steady state probability density of the optimal portfolio is uniform for a single stock but generally blows up even in the simple case of two uncorrelated stocks.

Key words. transaction costs, general utility function, asymptotics, cost minimization, oblique boundary conditions, portfolio optimization

AMS subject classifications. 91G80, 91G10, 35K20, 35R35

DOI. 10.1137/07070334X

1. Introduction.

Simple trading strategies developed by Merton, Black, and Scholes [1] neglect the effects of transaction costs and other market frictions. This is a reasonable first approximation, so it makes sense to study market frictions as perturbations of these simpler idealized theories. For this paper, we consider the case of proportional transaction costs (see, for example, Magill and Constantinides [2]), where we lose a small fixed fraction of the value of each trade. This is a common model for approximating losses due to the "bid-ask" spread in stock prices. The (approximate) optimal strategy keeps the portfolio inside a *hold region*, $\mathcal{H}$, centered about the idealized portfolio position. No trading occurs in the interior of this hold region. Trades occur only when the portfolio's position is on the boundary of $\mathcal{H}$ to prevent the portfolio from leaving $\mathcal{H}$. We study the asymptotic size and shape of the hold region for small transaction costs.

We analyze the asymptotic behavior of the optimal hold region in several related ways. For the case of a single stock we give a simple heuristic argument for the size of the hold region as a function of the transaction cost proportion parameter, ε. Our derivation relies on dividing the effect of the transaction cost into the *trading cost*, which is the average cost of the trades per unit time, and the *opportunity cost*, which is how much the portfolio underperforms the ideal portfolio due to deviating from the ideal portfolio's position. Expanding the hold region decreases the trading cost but increases the opportunity cost. Balancing these two factors leads to the optimal hold region. It turns out that the quasi-steady state probability density for the portfolio position, which we call u, is constant within the hold region.

*Received by the editors September 20, 2007; accepted for publication (in revised form) December 7, 2009; published electronically February 26, 2010.
http://www.siam.org/journals/siap/70-6/70334.html
†Courant Institute of Mathematical Sciences, New York University, 251 Mercer St., New York, NY 10012-1185 (goodman@cims.nyu.edu).
‡Department of Mathematics, Santa Clara University, 500 El Camino Real, Santa Clara, CA 95053 (dostrov@scu.edu).

This seemingly heuristic analysis extends to a multiple stock context. We justify it by relating it to the presumably more rigorous picture of perturbation theory applied to f, the value function for the expected utility of the portfolio at a later final time. We show that our heuristic probability-based formulation is dual to the value function formulation in the sense of the duality theory of optimal design problems governed by PDEs [3]. In particular, the perturbative correction to the value function for the expected utility (namely, f_4 below) is the adjoint field to the probability density, u, and vice versa.

We believe that the (approximate) steady state distribution, u, is interesting and useful. In subsequent work [4], we use it to derive asymptotically optimal trading strategies for stocks and liquid options. Here, we give the surprising result that u need not be constant in the multistock case, even when the stocks are uncorrelated. The hold region in the uncorrelated case is an n-dimensional box (see also [5]). But even in the case of two uncorrelated stocks, the steady state probability density generically blows up in two of the four corners of the rectangular hold region. This may seem paradoxical, as the transaction cost is determined by integrating the probability density along the boundary of the hold region, which would seem to discourage having an infinite probability density there. However, it turns out that the transaction cost rate integral remains finite.

The case of proportional transaction costs has a history of approaches focusing on f, the value function for expected utility. For a portfolio with a single stock, the problem has been studied by Shreve and Soner [6] and Janeček and Shreve [7], who applied viscosity solution methods (see Davis et al. [8]) to the case of power law and logarithmic utility functions. Whalley and Wilmott [9] employed asymptotic analysis to the case of exponential utility functions. Both groups found that transaction costs led to a minimal loss on the order of $\varepsilon^{2/3}$ in the expected utility. (Note that an initial transaction to shift the portfolio to optimal balance or a final transaction to liquidate the portfolio create ε order losses and therefore, being higher order than $\varepsilon^{2/3}$, are irrelevant to this result.) They also found that the optimal trading strategy allowed the portfolio to move freely within the interior of the one-dimensional hold region. The hold region extends a distance γ, which is proportional to $\varepsilon^{1/3}$, in either direction about its center, located at the idealized Merton strategy. For portfolios with n uncorrelated stocks, Atkinson and Mokkhavesa [5] showed that the hold region is an n-dimensional rectangular box. In the case of correlated stocks, numerical computations by Muthuraman and Kumar [10] and Atkinson and Ingpochai [11] suggest that the hold region is a smoothly distorted rectangular box.

Our work tries to explain these earlier results by studying the trading strategies they imply. In section 2, we consider the single stock case and give an intuitive, but informal, argument showing that for a general utility function the leading order loss in the expected utility due to the presence of transaction costs is an explicit trade-off between the opportunity cost, which is of order γ^2, and the trading cost, which is of order ε/γ. Optimizing $\gamma^2 + \epsilon/\gamma$ with respect to γ immediately yields the order $\varepsilon^{1/3}$ size of γ and the order $\varepsilon^{2/3}$ loss in f. Keeping track of the coefficients in front of these two loss terms, we will get full expressions for the radius γ and the loss in f, which match the results of both Shreve's group and Whalley and Wilmott, generalized to any utility function. They also match the more recent results of Mokkhavesa and Atkinson in [12].[1] As in these other papers, much of our analysis centers on the key

[1]We note that in [6, 7] and [11, 12], consumption occurs continuously over an infinite time horizon, while in [9] and our paper, consumption occurs only at the end of a finite time horizon. Either consumption model yields similar results.

imbalance variable

$$\xi(t) = X(t) - m(Z(t), t),$$

which measures the difference between $X(t)$, the book value of the stock in the portfolio, and m, the idealized Merton value that X optimally takes in the absence of transaction costs (where $Z(t)$ is the total book value, both in stock and cash, of the portfolio). With this notation, the hold region is simply expressed by $\xi \in (-\gamma, \gamma)$.

In section 3, we extend and justify this intuitive result in a multiple stock setting. We begin by generalizing our heuristic arguments to multiple dimensions, which yields multistock equations for the probability density, u, and for the opportunity and trading costs. We review the more traditional approach of constructing an asymptotic expansion for f, the value function for the expected utility:

$$f(z, \xi, t, \varepsilon) \sim f^0 + \varepsilon^{1/3} f^1 + \varepsilon^{2/3} f^2 + \varepsilon f^3 + \varepsilon^{4/3} f^4.$$

By analyzing the conditions on the boundary of the hold region, we conclude that f^0 through f^3 solely depend upon t and the portfolio's book value, z, while f^4 depends upon ξ as well as z and t. Therefore, f^4 is the key term in the expansion. We determine the PDE and boundary conditions governing f^4, which parallel the results of Atkinson and Mokkhavesa in [5]. Finally, by computing the adjoint of the heuristic equations for the probability density, u, we show that minimizing the sum of our heuristic expressions for the opportunity and trading costs leads to a Lagrange multiplier PDE and boundary conditions that are identical to the f^4 PDE and boundary conditions. This establishes the equivalence of the two approaches and justifies the heuristics used to determine both our equations governing u and our expressions for the leading order loss in the expected utility as the sum of opportunity and trading costs.

In the multiple stock case, we are generally unable to explicitly determine the hold region or the functions u and f. However, in section 4 we are able to use the equations for u to show that, unlike in the single stock case, u, generically, is far from uniform. This relies on qualitative results for solutions of diffusion equations with oblique boundary conditions in parallelogram regions.

More detailed explanations of some results in this paper, including single stock versions of the arguments in section 3, can be found in the working paper [13].

2. Analysis for single stock portfolios.

2.1. Merton case (no transaction costs). We begin by reviewing Merton's optimal allocation within a portfolio of cash and stock where no transaction costs are incurred for buying or selling stock. If we know that our portfolio is worth a total of z dollars at time t, we seek a trading strategy for the stock at subsequent times that maximizes the expected utility of $Z(T)$, the total worth of the portfolio, at a later time T. We will let f denote the maximum expected utility; that is,

$$f(z, t) = \sup \left(E_{z,t} \left[U \left(Z(T) \right) \right] \right). \tag{1}$$

We assume that the cash can be invested at the risk-free rate r and that the stock moves by geometric Brownian motion with mean μ and volatility σ. Therefore, for $X(t)$, the dollar worth of stock in our portfolio at time t, and $Y(t)$, the worth of the cash in our portfolio at time t, we have

$$dX(t) = \mu X(t) dt + \sigma X(t) dB(t), \tag{2}$$

$$dY = rY dt, \tag{3}$$

where $B(t)$ is a Brownian motion. Since the total worth of the portfolio

$$Z(t) = X(t) + Y(t),$$

we have

$$dZ(t) = dX(t) + dY(t) = [(\mu - r)\, X(t) + rZ(t)]\, dt + \sigma X(t) dB(t). \tag{4}$$

Since trading is free, our trading strategy for the stock is based solely on $Z(t)$ and t; that is, $X(t) = x(Z(t), t)$ for some trading strategy function x, and so the Hamilton–Jacobi–Bellman equation for $f(Z, t)$ is

$$0 = \sup_x \left\{ f_t + [(\mu - r)\, x + rZ]\, f_z + \frac{1}{2}\sigma^2 x^2 f_{zz} \right\}. \tag{5}$$

We define the Merton value, $m(Z, t)$, to be the value of x that maximizes the right-hand side of this equation. This is determined by setting the derivative with respect to x of the braced term in (5) equal to zero, which yields the formula for the Merton value:

$$m(Z(t), t) = -\frac{(\mu - r)\, f_z(Z(t), t)}{\sigma^2 f_{zz}(Z(t), t)}. \tag{6}$$

Note that m is positive since $f_z > 0$ and $f_{zz} < 0$, which can be established using the fact that $U' > 0$ and $U'' < 0$ for all utility functions.

2.2. Heuristic discussion of the transaction costs case. Now we consider the effect of adding small proportional transaction costs to this model. Let $L(t)$ be the dollar amount of cash spent buying stock up to time t, and $M(t)$ be the dollar worth of all stock sold up to time t. For the proportional transaction cost model with a transaction parameter ε and transaction constants b and c, we update (2) and (3), the differentials of X and Y, to incorporate L and M:

$$dX(t) = \mu X(t) dt + \sigma X(t) dB(t) + (1 - \varepsilon b) dL(t) - dM(t), \tag{7}$$
$$dY(t) = rY(t) dt + (1 - \varepsilon c) dM(t) - dL(t). \tag{8}$$

Notice that a unit increase in L removes a unit of Y but adds only $(1 - \varepsilon b)$ units to X; that is, there is a loss of εb dollars for every dollar spent buying stock. Similarly, selling a dollar of stock removes a dollar from X but adds only $(1 - \varepsilon c)$ dollars to Y for a loss of εc dollars.

As in the Merton analysis, we use $Z(t) = X(t) + Y(t)$, which is the book dollar value of the portfolio at time t. If the portfolio were liquidated, it would produce only $Z - \varepsilon c X$ of cash. We now introduce the key variable ξ to denote the difference between the current stock position and the ideal Merton stock position, $m(Z, t)$; that is, we define

$$\xi(t) = X(t) - m(Z(t), t). \tag{9}$$

The analysis of the transaction case is significantly simplified by using (z, ξ), as opposed to (x, y), coordinates. Combining (7) through (9) and applying the Ito calculus shorthand $(dB)^2 = dt$ gives the differentials of Z and ξ:

$$dZ = [(\mu - r)(m + \xi) + rZ]\, dt + \sigma(m + \xi)dB - \varepsilon(bdL + cdM), \tag{10}$$

$$\begin{aligned} d\xi &= dX - m_z dZ - m_t dt - \frac{1}{2} m_{zz} (dZ)^2 \\ &= \begin{bmatrix} \mu (m + \xi) - m_t \\ -m_z ((\mu - r)(m + \xi) + rz) \\ -\frac{1}{2} m_{zz} \sigma^2 (m + \xi)^2 \end{bmatrix} dt \\ &\quad + \sigma (1 - m_z)(m + \xi) dB \\ &\quad + (1 - \varepsilon b(1 - m_z))\, dL - (1 - \varepsilon c m_z)\, dM. \end{aligned} \tag{11}$$

We propose a probabilistic way to estimate both the optimal trading strategy and the expected loss in maximum utility due to trading costs when the trading costs are small. As stated in the introduction, after an immediate initial trade, we expect the optimal trading strategy to be a singular control that operates to keep the portfolio within a certain narrow region near the Merton portfolio. We call the inside of this narrow region the *hold region*, $\mathcal{H}$, since we do not trade when we are within it. In the present variables, this means keeping ξ, the deviation from the Merton balance, small. In particular, we assume in this section that there is a small $\gamma(z, t) > 0$ so that $dL \neq 0$ only when $\xi = -\gamma$, and $dM \neq 0$ only when $\xi = \gamma$; that is, we trade only on the boundary of the hold region, $\xi \in (-\gamma, \gamma)$. The more rigorous PDE-based asymptotic expansion in section 3.2 will validate our assumption here that, to leading order, the hold region is a symmetric interval of ξ values centered about the Merton value, $\xi = 0$.

Given that the hold region for ξ should be small and slowly varying, we expect that ξ is roughly in stochastic equilibrium within this region. That is, if we imagine Z and t as fixed, the one-dimensional process $\xi(t)$ will have a probability distribution that is (approximately) an equilibrium for its stochastic differential equation with reflecting boundary conditions at $\xi = \pm\gamma$.

To determine this equilibrium distribution for small γ, we make several simplifying approximations to (11). First, we neglect the drift terms—that is, all of the dt terms—which should have a small effect on the steady state, since ξ stays within a small domain. Second, we neglect ξ in $(m+\xi)dB$ because ξ should be much smaller than m. Finally, we neglect the $O(\varepsilon)$ terms. This leaves us with the simple expression

$$d\xi \approx dL - dM + adB, \tag{12}$$

where a is given by

$$a = \sigma (1 - m_z)\, m. \tag{13}$$

Since $dL \neq 0$ only when $\xi = -\gamma$, $dM \neq 0$ only when $\xi = \gamma$, and L and M are nondecreasing, the differential equation corresponding to (12) for the equilibrium probability density, $u(\xi)$, is

$$Au_{\xi\xi} = 0, \tag{14}$$

subject to the Neumann boundary conditions,

$$u_\xi = 0 \quad \text{at } \xi = \pm\gamma, \tag{15}$$

where A, which we assume to be nonzero, is defined by

$$A = \frac{1}{2} a^2. \tag{16}$$

Here, unlike the multiple stock case in section 3, we can divide (14) by A and solve (14) and (15), which establishes that the probability density function is uniform in the hold region:

$$u(\xi) = \frac{1}{2\gamma}.$$

Let $f(z,t)$ be the idealized Merton value function in (1). We want to know how much the optimal transaction cost strategy underperforms the Merton zero transaction cost strategy. We assume the heuristic that the maximum expected utility is affected by small transaction costs primarily through the variable Z and not through direct changes to f. We consider the effect of direct changes to f in section 3, which will justify this heuristic. Under this heuristic, we calculate

$$df = f_t dt + f_z dZ + \frac{1}{2} f_{zz} (dZ)^2,$$

with $(dZ)^2 = \sigma^2 (m+\xi)^2 dt$. Using (10), this leads to

$$df = \left[\begin{array}{c} f_t + f_z \left(rZ + (\mu - r)m + (\mu - r)\xi\right) \\ + f_{zz}\left(\frac{\sigma^2 m^2}{2} + \sigma^2 m\xi + \frac{\sigma^2\xi^2}{2}\right) \end{array} \right] dt \tag{17}$$
$$+ f_z \sigma (m+\xi) dB - \varepsilon f_z (b dL + c dM).$$

Since f satisfies the Merton equation

$$f_t + \left[(\mu - r)\, m + rZ\right] f_z + \frac{1}{2}\sigma^2 m^2 f_{zz} = 0,$$

the df expression in (17) simplifies to

$$df = \left[(\mu - r)\xi f_z + \sigma^2 m \xi f_{zz} + \frac{\sigma^2 \xi^2}{2} f_{zz}\right] dt \tag{18}$$
$$+ \sigma(m+\xi) f_z dB - \varepsilon f_z (b dL + c dM).$$

We next look at estimating the optimal $\gamma(z,t)$. We start with (18) and take the approximate equilibrium (with respect to ξ) expectation, $E[\cdot]$, and, since our uniform density function implies that $E[\xi] = 0$, we have

$$E[df] = \frac{\sigma^2 f_{zz}}{2} E\left[\xi^2\right] dt - \varepsilon f_z \left(b E[dL] + c E[dM]\right). \tag{19}$$

Recalling that $f_z > 0$ and $f_{zz} < 0$, we can see that choosing γ to minimize our transaction loss—that is, choosing γ to maximize the negative quantity $E[df]$—requires a balance between the two terms on the right-hand side of (19). The first term on the right-hand side is the *opportunity cost*, which gives the loss due to the portfolio's deviation from the optimal Merton balance. Since $u(\xi) = \frac{1}{2\gamma}$,

$$E\left[\xi^2\right] = \int_{-\gamma}^{\gamma} \xi^2 u(\xi) d\xi = \frac{1}{2\gamma}\int_{-\gamma}^{\gamma} \xi^2 d\xi = \frac{\gamma^2}{3}, \tag{20}$$

so we see how the opportunity cost decreases as we reduce γ. The second term on the right-hand side of (19) is the *trading cost*, which gives the loss due to trading on the boundaries $\xi = \pm\gamma$. Since this cost is proportional to the density of ξ on the boundaries, it is inversely proportional to γ, and so we see how the trading cost is increased by reducing γ.

More precisely, to evaluate $E[dL]$ and $E[dM]$ in the trading cost, we start by taking the expected value of the Ito chain rule formula applied to ξ^2:

$$E\left[d\left(\xi^2\right)\right] = 2E\left[\xi d\xi\right] + E\left[(d\xi)^2\right]. \tag{21}$$

From our assumption of approximate equilibrium, we have $\frac{d}{dt}E\left[\xi^2\right] = 0$, so the left-hand side of (21) vanishes. To evaluate the leading order behavior of each of the two terms on the right-hand side of (21), we apply (12) to find that

$$E\left[(d\xi)^2\right] = a^2 dt$$

and

$$E\left[\xi d\xi\right] = E[\xi dL] - E[\xi dM].$$

As noted above, $dL \neq 0$ only when $\xi = -\gamma$, and $dM \neq 0$ only when $\xi = \gamma$. Therefore $E[\xi dL] = -\gamma E[dL]$ and $E[\xi dM] = \gamma E[dM]$. Since symmetry requires that $E[dL] = E[dM]$, we have from (16) and (21) that

$$E[dL] = E[dM] = \frac{A}{2\gamma}dt. \tag{22}$$

Now we can evaluate $E[df]$ by substituting (20) and (22) into (18), which yields

$$\frac{E[df]}{dt} = \frac{\sigma^2 f_{zz}\gamma^2}{6} - \varepsilon f_z(b+c)\frac{A}{2\gamma}. \tag{23}$$

Maximizing $\frac{E[df]}{dt}$ over γ leads to the optimal width of the hold region:

$$\gamma = \varepsilon^{\frac{1}{3}}\left(-\frac{3}{2}\frac{A(b+c)f_z}{\sigma^2 f_{zz}}\right)^{\frac{1}{3}}. \tag{24}$$

Inserting (24) back into (23) yields the optimal loss rate:

$$\frac{E[df]}{dt} = \varepsilon^{\frac{2}{3}}\frac{\left(\sigma^2 f_{zz}\right)^{\frac{1}{3}}}{2}\left(\frac{3}{2}A(b+c)f_z\right)^{\frac{2}{3}}. \tag{25}$$

These final two formulas, (24) and (25), will be corroborated by connecting the heuristic results used to justify them to a more rigorous free boundary PDE asymptotic expansion argument. This is done in a multidimensional context in the next section.

We remark that b and c (the buy and sell transaction cost parameters, respectively) occur in (25) only in their sum, $b+c$. This is the transaction cost for a "round trip," buying and then selling the same share of stock. The Merton strategy involves constantly buying and selling small amounts of stock to compensate for small market movements, so it is natural that the round trip cost is what matters. This is why proportionate transaction cost is a model of the bid-ask spread, which also is the cost of a round trip.

3. Analysis for multiple stock portfolios. We now consider portfolios with n stocks—as opposed to a single stock—and cash.

Important summation convention. Throughout this section we employ the following slightly nonstandard summation convention: Unless otherwise stated, for any term on the right-hand side of an equation, we sum from 1 to n over all indices (i, j, k, l, m) that appear in the term unless the index also appears on the left-hand side of the equation (in which case we do not sum over that index).

For example, the total worth of the portfolio is now

$$Z(t) = X_i(t) + Y(t)$$

since our summation convention requires that we sum the $X_i(t)$ term from $i = 1$ to n (where X_i is the book worth of stock i in the portfolio).

Generalizing the Merton analysis in section 2.1 (see [1] or [13]) yields for each stock i that m_i, the ideal selected value of stock i in the absence of transaction costs, is

$$m_i(Z(t), t) = -\frac{\sigma_{ji}^{-1}\sigma_{jk}^{-1}(\mu_k - r)\, f_z(Z(t), t)}{f_{zz}(Z(t), t)}, \tag{26}$$

where μ_i and σ_{ij} are the components of the constant expected return vector and the volatility matrix for the geometric Brownian motion of the stocks. Here, for the sole term on the right-hand side, we sum j and k from 1 to n, but not i since i appears on the left-hand side of the equation.

3.1. Heuristic discussion of the transaction costs case. This section formulates the asymptotic optimal trading problem as a PDE-constrained optimal domain problem. For any given trial hold region, $\mathcal{H}$, there is a PDE to solve, (43) below, that determines the steady-state probability density of the imbalance. This PDE comes with oblique boundary conditions on $\partial\mathcal{H}$. The problem is to find $\mathcal{H}$ so that an objective function is minimized. This objective function (36) is a sum of the opportunity loss rate, which is an integral over $\mathcal{H}$, and the transaction cost rate, which is an integral over $\partial\mathcal{H}$. Our discussion here is heuristic, but it agrees with the more systematic arguments of section 3.2, as we will establish in section 3.3.

Generalizing the single stock notation of section 2, let $L_i(t)$ be the dollar amount of cash spent buying stock i up to time t, and $M_i(t)$ be the dollar worth of all stock i sold up to time t. For transaction parameter ε and transaction constants b_i and c_i, the differentials of X_i and Y are now

$$dX_i(t) = \mu_i X_i(t)dt + X_i(t)\sigma_{ij}dB_j + (1 - \varepsilon b_i)dL_i(t) - dM_i(t), \tag{27}$$

$$dY(t) = rY(t)dt + (1 - \varepsilon c_i)dM_i(t) - dL_i(t). \tag{28}$$

As before, the key imbalance variables ξ_i denote the difference between $X_i(t)$, the book worth of the portfolio's position in stock i, and $m_i(Z, t)$, the book worth of the portfolio's ideal Merton position in stock i:

$$\xi_i(t) = X_i(t) - m_i(Z(t), t). \tag{29}$$

Also, as before, we use (z, ξ_i), as opposed to (x_i, y), coordinates. Combining (27) through (29) and applying the Ito calculus shorthand,

$$dB_i dB_j = \begin{cases} dt & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

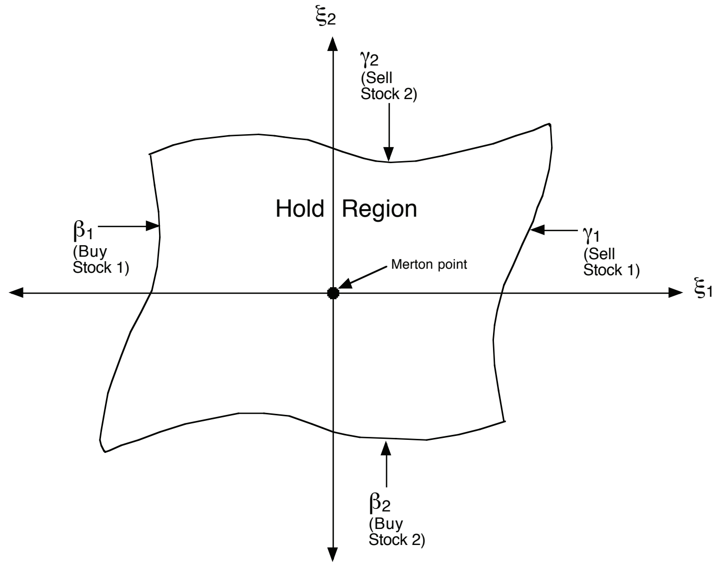

FIG. 1. *The hold region (also called the no transaction region) centered about the Merton portfolio.*

gives the differentials of Z and ξ_i:

$$dZ = [(\mu_i - r)(m_i + \xi_i) + rZ]\,dt + [(m_i + \xi_i)\sigma_{ij}]\,dB_j - \varepsilon\,(b_i dL_i + c_i dM_i)\,, \tag{30}$$

$$\begin{aligned} d\xi_i &= dX_i - (m_i)_t\,dt - (m_i)_z\,dZ - \frac{1}{2}(m_i)_{zz}(dZ)^2 \\ &= \begin{bmatrix} \mu_i\,(m_i + \xi_i) - (m_i)_t \\ -(m_i)_z\,((\mu_j - r)(m_j + \xi_j) + rz) \\ -\frac{1}{2}(m_i)_{zz}\,(m_j + \xi_j)\sigma_{jk}\sigma_{lk}(m_l + \xi_l) \end{bmatrix} dt \\ &\quad + [(m_i + \xi_i)\,\sigma_{ik} - (m_i)_z\,(m_j + \xi_j)\,\sigma_{jk}]\,dB_k \\ &\quad + (1 - \varepsilon b_i)\,dL_i + \varepsilon\,(m_i)_z\,b_j dL_j - dM_i + \varepsilon\,(m_i)_z\,c_j dM_j. \end{aligned} \tag{31}$$

We expect that the optimal trading strategy involves a small hold region, $\mathcal{H}$. The trading strategy is to trade only on $\partial\mathcal{H}$ so that the portfolio never leaves $\mathcal{H}$. Given that ξ is small in $\mathcal{H}$ and that $\mathcal{H}$ is smoothly varying in time, suppose that the distribution of ξ in $\mathcal{H}$ is always approximately in statistical steady state. We determine $\mathcal{H}$ at each t to optimize this steady state. The boundary of $\mathcal{H}$ consists of $2n$ smooth intersecting surfaces: β_i, where we buy stock i, and γ_i, where we sell stock i. (See Figure 1 for the two stock case.)

Following the same heuristic assumptions for df used in section 2.2, we get the multidimensional form of (19):

$$E[df] = \frac{f_{zz}}{2} E\,[\xi_i \sigma_{ik} \sigma_{jk} \xi_j]\,dt - \varepsilon f_z\,(b_i E\,[dL_i] + c_i E\,[dM_i])\,. \tag{32}$$

As before, the first term on the right-hand side of (32) is the *opportunity cost*, which gives the loss due to the portfolio's deviation from the optimal Merton balance, and the second term on the right-hand side is the *trading cost*, which gives the loss due to trading on the boundary of the hold region.

The analysis of the trading cost term with multiple stocks requires a transformation so that we can eventually apply the single variable analysis. Consider a generic point, which we will call $\overline{\xi}$, on β_i, the boundary where we buy stock i. We are interested in the expected ΔL_i due to contact with β_i near this point over a small time interval, Δt. Because Δt is small, $\xi(t)$ will touch $\partial\mathcal{H}$ almost entirely on β_i near $\overline{\xi}$. Given this, if we assume that β_i is smooth at $\overline{\xi}$ and that the density is continuous at $\overline{\xi}$, we can model ΔL_i near $\overline{\xi}$ by approximating the hold region as the half space bounded by the $(n-1)$-dimensional plane tangent to β_i at $\overline{\xi}$. Let $\nu = (\nu_1, \ldots, \nu_n)$ be the outward normal to $\partial\mathcal{H}$ at $\overline{\xi}$. Our argument replaces the probability density by its value at $\overline{\xi}$, which is $u(\overline{\xi})$.

Now with i fixed, we look at the differential of each of the ξ_l in (31). We make the same simplifying assumptions as we had in the single variable case: specifically, we neglect the drift terms (i.e., all of the dt terms) because the hold region is small, we neglect ξ_j in $(m_j + \xi_j)$ because ξ_j should be much smaller than m_j, and we neglect the $O(\varepsilon)$ terms, which leaves us with

$$d\xi_l \approx dL_l - dM_l + a_{lk} dB_k, \tag{33}$$

where

$$a_{lk} = m_l \sigma_{lk} - (m_l)_z \, m_j \sigma_{jk}. \tag{34}$$

We may neglect all of the dM_l because $\overline{\xi}$ is not near a sell boundary. We may neglect all of the dL_l except dL_i because $\overline{\xi}$ is far from β_l for $l \neq i$. Therefore, $\nu_i dL_i \approx \nu_l \left(dL_l - dM_l\right)$.

Now define

$$D_i(\xi) = -\nu_l \left(\xi_l - \overline{\xi}_l\right),$$

which is the distance from ξ to the tangent plane for β_i at $\overline{\xi}$. Differentiating gives $dD_i = -\nu_l d\xi_l$. Substituting (33) for $d\xi_l$ yields

$$dD_i = -\nu_i dL_i + \widehat{a} dB, \tag{35}$$

with B being a scalar Brownian motion defined by $\widehat{a} dB = -\nu_l a_{lk} dB_k$, where $\widehat{a}^2 = \nu_j a_{jk} a_{lk} \nu_l$. The argument in section 2.2 applies to this *scalar* reflected diffusion. There is a similar argument for dM_i near one of the sell boundaries.

With all this (see [13] for more details), the differential loss of expected utility (32) is the multidimensional generalization of (23):

$$\begin{aligned} E[df] = {} & \frac{f_{zz}}{2} \left(\int_{\mathcal{H}} \xi_i \sigma_{ik} \sigma_{jk} \xi_j u(\xi) d\xi \right) dt \\ & + \varepsilon f_z A_{jl} \left(b_i \int_{\beta_i} \frac{\nu_j(\xi)\nu_l(\xi)}{\nu_i(\xi)} u(\xi)\, dS \right. \\ & \left. \qquad - c_i \int_{\gamma_i} \frac{\nu_j(\xi)\nu_l(\xi)}{\nu_i(\xi)} u(\xi)\, dS \right) dt, \end{aligned} \tag{36}$$

where

$$A_{ij} = \frac{1}{2} a_{ik} a_{jk} \tag{37}$$

is a symmetric positive semidefinite matrix. The integrals with dS are with respect to $(n-1)$-dimensional surface measure on the buy and sell surfaces. Also note that ν_i is negative on any β_i and positive on any γ_i, so, since $f_{zz} < 0$ and $f_z > 0$, we have that both the opportunity cost and the trading cost are negative quantities.

We write the PDE and boundary conditions satisfied, to leading order, by the steady state probability density $u(\xi)$ in $\mathcal{H}$. We review this basic duality argument to establish notation that we will use again in section 3.3. The backward equation corresponding to the process (33) is

$$f_t = -A_{ij} f_{\xi_i \xi_j}, \qquad \xi \in \mathcal{H}, \tag{38}$$

subject to the Neumann boundary condition

$$f_{\xi_k} = 0, \qquad \xi \in \beta_k \cup \gamma_k, \tag{39}$$

where $k = 1, 2, \ldots, n$. Even if the probability density for $\xi(t)$ is time dependent, $u(\xi, t)$, we have the basic conservation of expected value:

$$0 = \frac{d}{dt} \int_{\mathcal{H}} u f d\xi. \tag{40}$$

Using (38) and (39) and integrating by parts, this leads to

$$\begin{aligned} 0 &= \int_{\mathcal{H}} u_t f + u f_t d\xi \\ &= \int_{\mathcal{H}} u_t f - u A_{ij} f_{\xi_i \xi_j} d\xi \\ &= \int_{\mathcal{H}} \left(u_t - A_{ij} u_{\xi_i \xi_j} \right) f d\xi && (41) \\ &\quad + A_{ij} \int_{\partial \mathcal{H}} \left(u_{\xi_i} \nu_j f - u \nu_j f_{\xi_i} \right) dS. && (42) \end{aligned}$$

Here f is arbitrary, so the volume integral (41) and the surface integral (42) must vanish separately. The volume part (41) gives, if u is independent of t,

$$0 = A_{ij} u_{\xi_i \xi_j}, \qquad \xi \in \mathcal{H}. \tag{43}$$

We study the surface integral,

$$0 = A_{ij} \int_{\partial \mathcal{H}} \left(u_{\xi_i} \nu_j f - u \nu_j f_{\xi_i} \right) dS, \tag{44}$$

by integrating by parts to put the ξ derivatives on u. This is subtle because $\partial\mathcal{H}$ is a collection of possibly curved surfaces.

The integral over $\partial\mathcal{H}$ is the sum of integrals over the individual buy and sell surfaces β_k and γ_k. We parametrize the surfaces β_k and γ_k by the $n-1$ component variable $\widehat{\xi}_k = (\xi_1, \xi_2, \ldots, \xi_{k-1}, \xi_{k+1}, \ldots, \xi_n)$. Let B_k and Γ_k be the respective orthogonal projections of β_k and γ_k onto $\widehat{\xi}_k$-space. Therefore, a surface integral over β_k may be expressed as an ordinary integral $\int_{B_k} \cdots d\widehat{\xi}_k$. We assume that β_k and γ_k are the graphs of functions $\widehat{\beta}_k(\widehat{\xi}_k)$ and $\widehat{\gamma}_k(\widehat{\xi}_k)$, respectively. The surfaces β_k and γ_k also may be described as zero level surfaces, over B_k and Γ_k, of functions

$$\phi_k(\xi) = \widehat{\beta}_k\left(\widehat{\xi}_k\right) - \xi_k, \qquad \psi_k(\xi) = \widehat{\gamma}_k\left(\widehat{\xi}_k\right) - \xi_k. \tag{45}$$

We develop a notation for differentiation within the surfaces β_k and γ_k. Let $g(\xi)$ be a generic function. If we are restricted to β_k, the operator $d_{k,i}$ will denote the "total" derivative of g in the ξ_i direction. If $i \neq k$, this is

$$d_{k,i}g \;=\; \partial_{\xi_i} g(\widehat{\xi}_k, \widehat{\beta}_k(\widehat{\xi}_k)) \;=\; g_{\xi_i} \;+\; \left(\widehat{\beta}_k\right)_{\xi_i} g_{\xi_k} \;=\; g_{\xi_i} \;+\; (\phi_k)_{\xi_i} \, g_{\xi_k}. \tag{46}$$

We also use the last equality to define $d_{k,k}$, giving $d_{k,k}g = 0$ because $(\phi_k)_{\xi_k} = -1$. Similarly, if we are restricted to γ_k, we define $d_{k,i}g = g_{\xi_i} + (\psi_k)_{\xi_i} \, g_{\xi_k}$.

The key point is that we can apply integration by parts to the $d_{k,i}$ operators when we express the surface integrals over β_k and γ_k in terms of ordinary integrals over B_k and Γ_k.

The elements of integration on β_k and γ_k satisfy $\nu_i dS = (\phi_k)_{\xi_i} \, d\widehat{\xi}_k$ and $\nu_i dS = -\,(\psi_k)_{\xi_i} \, d\widehat{\xi}_k$, where we do not sum over k because the left-hand sides implicitly depend on k. Therefore, we can rewrite the boundary integral equation (44) as (summing now over k)

$$\begin{aligned} 0 = A_{ij} \Bigg[&\int_{B_k} \left(u_{\xi_i} \, (\phi_k)_{\xi_j} \, f \;-\; u \, (\phi_k)_{\xi_j} \, f_{\xi_i} \right) d\widehat{\xi}_k \\ &- \int_{\Gamma_k} \left(u_{\xi_i} \, (\psi_k)_{\xi_j} \, f \;-\; u \, (\psi_k)_{\xi_j} \, f_{\xi_i} \right) d\widehat{\xi}_k \Bigg]. \end{aligned} \tag{47}$$

(Note that while ϕ_k and ψ_k are functions of ξ, the derivatives, $(\phi_k)_{\xi_j}$ and $(\psi_k)_{\xi_j}$, are functions only of $\widehat{\xi}_k$.) Now (46) and the Neumann boundary condition (39) imply that $d_{k,i}f = f_{\xi_i}$ on β_k or on γ_k. We substitute $d_{k,i}f$ for f_{ξ_i} in (47), and then we can integrate by parts, which yields

$$0 \;=\; A_{ij} \left(\begin{array}{c} \displaystyle\int_{B_k} \left[u_{\xi_i} \, (\phi_k)_{\xi_j} \;+\; d_{k,i} \left(u \, (\phi_k)_{\xi_j} \right) \right] f \, d\widehat{\xi}_k \\ \displaystyle -\int_{\Gamma_k} \left[u_{\xi_i} \, (\psi_k)_{\xi_j} \;+\; d_{k,i} \left(u \, (\psi_k)_{\xi_j} \right) \right] f \, d\widehat{\xi}_k \end{array} \right) \;+\; I^{n-2},$$

where I^{n-2} represents the $(n-2)$-dimensional integrals produced by the integration by parts on the boundaries of the sets B_k and Γ_k. With f now factored in the $(n-1)$-dimensional integrals over B_k and Γ_k, we have our desired oblique derivative boundary conditions on u:

$$\begin{aligned} 0 &= A_{ij} \left[u_{\xi_i} \, (\phi_k)_{\xi_j} \;+\; d_{k,i} \left(u \, (\phi_k)_{\xi_j} \right) \right] \qquad \text{on } B_k, \\ 0 &= A_{ij} \left[u_{\xi_i} \, (\psi_k)_{\xi_j} \;+\; d_{k,i} \left(u \, (\psi_k)_{\xi_j} \right) \right] \qquad \text{on } \Gamma_k, \end{aligned} \tag{48}$$

where, since we are on the sets B_k and Γ_k, we do not sum over k. Of course, there is a final integrability constraint on u since the integral of the probability over the hold region must equal 1:

$$\int_{\mathcal{H}} u(\xi) d\xi = 1. \tag{49}$$

For any given $\mathcal{H}$ bounded by buy and sell surfaces, optimal or not, the steady-state probability density for ξ is determined by the PDE (43) in the interior of $\mathcal{H}$, the boundary conditions (48) on $\partial\mathcal{H}$, and the integral constraint (49). The optimization problem is to find the surfaces β_k and γ_k that minimize the objective function (36) subject to these constraints. Section 3.2 presents a more formal asymptotic formulation of the problem. Section 3.3 shows that problem is dual to the one presented here, and hence equivalent to it.

3.2. Asymptotic analysis for multiple stocks. We now consider our problem from the point of view of a more systematic asymptotic approximation for the value function, f. This is the approach taken in most other work on transaction cost problems. Guided by our heuristic result (and previous papers) showing that the hold region size is $O\left(\varepsilon^{\frac{1}{3}}\right)$, we look for an expansion of f in powers of $\varepsilon^{\frac{1}{3}}$ and we scale ξ so that ξ from the previous subsection is now $\varepsilon^{\frac{1}{3}}\xi$ in this section. (We will revert back to section 3.1's scaling for ξ in section 3.3.) Therefore, the differentials dZ in (30) and $d\xi_i$ in (31) take the form

$$\begin{aligned} dZ &= \left[(\mu_i - r)\left(m_i + \varepsilon^{\frac{1}{3}}\xi_i\right) + rz\right] dt + \left[(m_i + \varepsilon^{\frac{1}{3}}\xi_i)\sigma_{ij}\right] dB_j \\ &\quad - \varepsilon\left(b_i dL_i + c_i dM_i\right), \\ d\xi_i &= dX_i - (m_i)_t\, dt - (m_i)_z\, dZ - \frac{1}{2}(m_i)_{zz}(dZ)^2 \\ &= \frac{1}{\varepsilon^{\frac{1}{3}}}\left\{\begin{array}{c} \left[\begin{array}{c} \mu_i\left(m_i + \varepsilon^{\frac{1}{3}}\xi_i\right) - (m_i)_t \\ -(m_i)_z\left((\mu_j - r)\left(m_j + \varepsilon^{\frac{1}{3}}\xi_j\right) + rz\right) \\ -\frac{1}{2}(m_i)_{zz}(m_j + \varepsilon^{\frac{1}{3}}\xi_j)\sigma_{jk}\sigma_{lk}(m_l + \varepsilon^{\frac{1}{3}}\xi_l) \end{array}\right] dt \\ + \left[a_{ik} + \varepsilon^{\frac{1}{3}}\alpha_{ik}\right] dB_k \\ + (1 - \varepsilon b_i)\, dL_i + \varepsilon (m_i)_z\, b_j dL_j - dM_i + \varepsilon (m_i)_z\, a_j dM_j \end{array}\right\}, \end{aligned}$$

where a_{ij} was defined in (34) and α_{ij} is defined by

$$\alpha_{ij} = -(m_i)_z \xi_k \sigma_{kj} + \xi_i \sigma_{ij}.$$

In the interior of the hold region where $dL_i = dM_i = 0$, these differentials lead to the following Hamilton–Jacobi–Bellman equation for $f(z, \xi, t) = \max\left\{E_{z,\xi,t}\left[U\left(z(T)\right)\right]\right\}$:

$$\begin{aligned} 0 &= f_t + \left[(\mu_j - r)\left(m_j + \varepsilon^{\frac{1}{3}}\xi_j\right) + rz\right] f_z \\ &\quad + \frac{1}{\varepsilon^{\frac{1}{3}}}\left[\begin{array}{c} \mu_i\left(m_i + \varepsilon^{\frac{1}{3}}\xi_i\right) - (m_i)_t - (m_i)_z\left((\mu_j - r)\left(m_j + \varepsilon^{\frac{1}{3}}\xi_j\right) + rz\right) \\ -\frac{1}{2}(m_i)_{zz}(m_j + \varepsilon^{\frac{1}{3}}\xi_j)\sigma_{jk}\sigma_{lk}(m_l + \varepsilon^{\frac{1}{3}}\xi_l) \end{array}\right] f_{\xi_i} \\ &\quad + \frac{1}{2}\frac{1}{\varepsilon^{\frac{2}{3}}}\left[\left(a_{ik} + \varepsilon^{\frac{1}{3}}\alpha_{ik}\right)\left(a_{jk} + \varepsilon^{\frac{1}{3}}\alpha_{jk}\right)\right] f_{\xi_i\xi_j} \\ &\quad + \frac{1}{\varepsilon^{\frac{1}{3}}}\left[\left(a_{ik} + \varepsilon^{\frac{1}{3}}\alpha_{ik}\right)\sigma_{jk}(m_j + \varepsilon^{\frac{1}{3}}\xi_j)\right] f_{\xi_i z} \\ &\quad + \frac{1}{2}\left[(m_i + \varepsilon^{\frac{1}{3}}\xi_i)\sigma_{ik}\sigma_{jk}(m_j + \varepsilon^{\frac{1}{3}}\xi_j)\right] f_{zz}. \end{aligned} \tag{50}$$

On the boundary of the hold region, the dL_k terms dominate the Hamilton–Jacobi–Bellman equation when we are on β_k, and the dM_k terms dominate when we are on γ_k. This yields the first derivative boundary conditions

$$0 = -\varepsilon b_k f_z + \frac{1}{\varepsilon^{\frac{1}{3}}} f_{\xi_k} - \varepsilon^{\frac{2}{3}} b_k f_{\xi_k} - \varepsilon^{\frac{2}{3}} b_k (m_j)_z\, f_{\xi_j} \qquad \text{on } \beta_k, \tag{51}$$

$$0 = \varepsilon a_k f_z + \frac{1}{\varepsilon^{\frac{1}{3}}} f_{\xi_k} - \varepsilon^{\frac{2}{3}} c_k (m_j)_z\, f_{\xi_j} \qquad \text{on } \gamma_k. \tag{52}$$

(We do not sum over k here.)

In the appendix we establish what some literature refers to as the "smooth pasting" condition. (See, for example, Dixit [14] and Dumas [15].) Specifically, we show that on the boundary of the hold region where $E[df]$ is optimized, we have a second derivative boundary condition that can be obtained by differentiating the first derivative boundary conditions in any direction transverse to the boundary. By choosing this direction to be ξ_k on β_k or γ_k, we get from (51) and (52) that the second derivative optimization condition is

$$0 = -\varepsilon b_k f_{z\xi_k} + \frac{1}{\varepsilon^{\frac{1}{3}}} f_{\xi_k\xi_k} - \varepsilon^{\frac{2}{3}} b_k f_{\xi_k\xi_k} - \varepsilon^{\frac{2}{3}} b_k \left(m_j\right)_z f_{\xi_j\xi_k} \qquad \text{on } \beta_k, \tag{53}$$

$$0 = \quad \varepsilon c_k f_{z\xi_k} + \frac{1}{\varepsilon^{\frac{1}{3}}} f_{\xi_k\xi_k} - \varepsilon^{\frac{2}{3}} c_k \left(m_j\right)_z f_{\xi_j\xi_k} \qquad \text{on } \gamma_k. \tag{54}$$

(Again, we do not sum over k here.)

Looking at the first order boundary conditions, (51) and (52), we see that the f_z term is $O\left(\varepsilon^{\frac{4}{3}}\right)$ smaller than the highest order term for f_{ξ_k}. This implies that ξ affects the solution for f only at the $O\left(\varepsilon^{\frac{4}{3}}\right)$ level, and so we consider an expansion for f of the form

$$f = f^0(z,t) + \varepsilon^{\frac{1}{3}} f^1(z,t) + \varepsilon^{\frac{2}{3}} f^2(z,t) + \varepsilon f^3(z,t) + \varepsilon^{\frac{4}{3}} f^4(z,\xi,t) + \cdots . \tag{55}$$

Given this expansion, the highest order terms in both the first and second derivative boundary conditions are $O\left(\varepsilon\right)$, which, specifically, from (51) and (52) are

$$\begin{aligned} \left(f^4\right)_{\xi_k} &= \quad b_k \left(f^0\right)_z \qquad \text{on } \beta_k, \\ \left(f^4\right)_{\xi_k} &= -c_k \left(f^0\right)_z \qquad \text{on } \gamma_k, \end{aligned} \tag{56}$$

and from (53) and (54) are

$$\left(f^4\right)_{\xi_k\xi_k} = 0 \qquad \text{on } \beta_k \text{ or } \gamma_k. \tag{57}$$

After inserting (55), the expansion for f, into (50), we collect terms with identical powers of $\varepsilon^{\frac{1}{3}}$, each of which we set to zero. The order ε^0 terms from this expansion lead, of course, to the Merton (no transaction costs) equation for f^0:

$$0 = \left(f^0\right)_t + \left(\left(\mu_i - r\right) m_i + rz\right) \left(f^0\right)_z + \frac{1}{2} m_i \sigma_{ik} \sigma_{jk} m_j \left(f^0\right)_{zz}.$$

The order $\varepsilon^{\frac{1}{3}}$ terms contain both f^0 and f^1 parts:

$$\begin{aligned} 0 = {} & \xi_j \left[\left(\mu_j - r\right)\left(f^0\right)_z + m_i \sigma_{ik}\sigma_{jk}\left(f^0\right)_{zz}\right] \\ & + \left[\left(f^1\right)_t + \left(\left(\mu_i - r\right) m_i + rz\right)\left(f^1\right)_z + \frac{1}{2} m_i \sigma_{ik}\sigma_{jk} m_j \left(f^1\right)_{zz}\right]. \end{aligned} \tag{58}$$

Substituting the expression for the Merton values, m_i, into (26), we see that the f^0 terms in (58) cancel each other, which leaves the Merton equation for f^1, but since the condition at T for f^1 is $f^1(z,T) = 0$, we must have, by uniqueness, that $f^1(z,t) = 0$. Knowing that $f^1 = 0$, we now collect the remaining order $\varepsilon^{\frac{2}{3}}$ terms:

$$\begin{aligned} 0 = {} & A_{ik}\left(f^4\right)_{\xi_i\xi_j} + \frac{1}{2}\xi_i \sigma_{ik}\sigma_{jk}\xi_j \left(f^0\right)_{zz} \\ & + \left(f^2\right)_t + \left(\left(\mu_i - r\right) m_i + rz\right)\left(f^2\right)_z + \frac{1}{2} m_i \sigma_{ik}\sigma_{jk} m_j \left(f^2\right)_{zz}, \end{aligned} \tag{59}$$

where the symmetric positive semidefinite matrix A_{ij} was defined in (37). Parallelling the single stock case, we assume that A_{ij} is positive definite, so we have an elliptic PDE in the ξ variables for f^4 of the form

$$K = A_{ij}\left(f^4\right)_{\xi_i\xi_j} + \frac{1}{2}\left(f^0\right)_{zz}\xi_i\sigma_{ik}\sigma_{jk}\xi_j, \tag{60}$$

where $-K$ is comprised of all three terms in the second line of (59), and we note that A_{ij}, K, σ, and $\left(f^0\right)_{zz}$ have no dependence on ξ.

To summarize: in the scaled ξ_i variables employed in this subsection, we see that f^4 must satisfy the elliptic PDE (60) within the hold region and, on the boundary of the hold region, f^4 must satisfy both the first order condition (56) and the second order optimality condition (57).

Generally, it is difficult to find a hold region and a function, f^4, that satisfies the elliptic PDE, (60), and the boundary conditions, (56) and (57). We note as an exception the case where $\sigma_{ij} = \sigma_i\delta_{ij}$, that is, the case where the n stocks are uncorrelated so that the matrix σ is diagonal with elements σ_i on the diagonal. In this case, it is easy to verify (see [5] or [13]) that the hold region in ξ-space is an n-dimensional box centered at the origin where $\tilde{\gamma}_i$, the half-width in the i direction, is given by

$$\tilde{\gamma}_i = \left(\frac{3}{2}A_{ii}\frac{(b_i+c_i)f_z^0}{\sigma_i^2 f_{zz}^0}\right)^{\frac{1}{3}}. \tag{61}$$

In other words, $-\hat{\beta}_i\left(\hat{\xi}_i\right) = \hat{\gamma}_i\left(\hat{\xi}_i\right) = \tilde{\gamma}_i$. Within this hold region, the function f^4 takes the form $f^4(z,\xi,t) = C_i^1(z,t)\,(\xi_i)^4 + C_i^2(z,t)\,(\xi_i)^2 + C_i^3(z,t)\,(\xi_i) + C^4(z,t)$, where $C_i^1(z,t) = -\frac{(b_i+c_i)f_z^0}{16(\tilde{\gamma}_i)^3}$, $C_i^2(z,t) = -6C_i^1(z,t)\,(\tilde{\gamma}_i)^2$, $C_i^3(z,t) = \frac{(b_i-c_i)f_z^0}{2}$, and the "constant" function $C^4(z,t)$ is indeterminable.[2]

For the single stock case in section 2.2, we see, after reconciling the two scalings for ξ, that our result for the half-width (61) verifies our expression for γ, the hold region radius, given in (24). In the next subsection, we will justify (23), our expression for $\frac{E[df]}{dt}$ as the sum of the trading cost and the opportunity cost. Since (25), our final expression for $\frac{E[df]}{dt}$, immediately follows, the results of section 2.2 will be justified.

3.3. Lagrange multiplier perspective. The less rigorous, but more intuitive, analysis of section 3.1 and the asymptotic expansions of section 3.2 are equivalent to each other. More precisely, they are related by the duality theory that is part of the Lagrange multiplier treatment of optimization problems with PDE constraints; see, e.g., [3]. We show that the Lagrange multiplier function that is dual to the probability constraint PDE (43) is equal to f_4, the leading order term depending on ξ in the maximum expected utility expansion from section 3.2.

We begin with the constrained optimization problem of section 3.1. We are interested in optimizing $E[df]$, given in (36), subject to the constraints on u of the PDE (43) within the body of the hold region, the Neumann condition (48) on the boundary of the hold region, and the integrability constraint (49). Recall that in section 3.1, the Neumann constraint (48) was derived and expressed in the projected spaces B_k and Γ_k so that integration by parts with the "total" derivative operator $d_{k,i}$

[2]Note that this form for f^4 determines K and therefore f^2. It can also be shown (see [13]) that f^4 is C^2 matching at the boundary, $\partial\mathcal{H}$, between the hold region and the trading region, and so f^4 is a classical solution since it is clearly C^2 in the interior of the hold region.

was possible. We will need to use these same techniques here, so we must express the boundary integral in our expression for $E[df]$ as integrals over the projected spaces B_k and Γ_k. This is accomplished by applying to the boundary integral in (36) the geometric facts that $\nu_i dS = (\phi_k)_{\xi_i}\, d\hat{\xi}_k$ (no sum over k) and $\frac{\nu_i}{\nu_k} = -(\phi_k)_{\xi_i}$ on each β_k and that $\nu_i dS = -(\psi_k)_{\xi_i}\, d\hat{\xi}_k$ (no sum over k) and $\frac{\nu_i}{\nu_k} = -(\psi_k)_{\xi_i}$ on each γ_k. This yields the form of function (36) that we want to optimize:

$$\frac{E[df]}{dt} = \frac{f_{zz}}{2}\left(\int_{\mathcal{H}} \xi_i \sigma_{ik}\sigma_{jk}\xi_j u(\xi) d\xi\right) - \varepsilon f_z A_{ij} \left(\begin{array}{c} b_k \int_{B_k} (\phi_k)_{\xi_i} (\phi_k)_{\xi_j}\, u(\xi) d\hat{\xi}_k \\ + c_k \int_{\Gamma_k} (\psi_k)_{\xi_i} (\psi_k)_{\xi_j}\, u(\xi) d\hat{\xi}_k \end{array} \right), \tag{62}$$

subject to the constraints (43), (48), and (49) on u.

We summarize a Lagrange multiplier optimization method explained in detail in [3] and Appendix 2 of [13]. We begin with the standard Lagrange multiplier expression for "the function to be optimized minus the inner product of the Lagrange multipliers with the constraints." Specifically, defining λ as the Lagrange multiplier function in the hold region corresponding to the PDE constraint (43), μ as the Lagrange multiplier function on the hold region's boundary corresponding to the Neumann constraint (48), and κ as the scalar Lagrange multiplier corresponding to the integrability constraint (49), this expression takes the form

$$\begin{aligned} \mathcal{L} = {} & \frac{f_{zz}}{2}\left(\int_{\mathcal{H}} \xi_i \sigma_{ik}\sigma_{jk}\xi_j u d\xi\right) \\ & - \varepsilon f_z A_{ij}\left(b_k \int_{B_k} (\phi_k)_{\xi_i} (\phi_k)_{\xi_j}\, u\, d\widehat{\xi_k} \;+\; c_k \int_{\Gamma_k} (\psi_k)_{\xi_i} (\psi_k)_{\xi_j}\, u\, d\widehat{\xi_k}\right) \\ & - A_{ij}\int_{\mathcal{H}} \lambda u_{\xi_i\xi_j} d\xi \;-\; A_{ij}\int_{B_k} \mu\left[u_{\xi_i} (\phi_k)_{\xi_j} + d_{k,i}\left(u\, (\phi_k)_{\xi_j}\right)\right] d\widehat{\xi_k} \\ & + A_{ij}\int_{\Gamma_k} \mu\left[u_{\xi_i} (\psi_k)_{\xi_j} + d_{k,i}\left(u\, (\psi_k)_{\xi_j}\right)\right] d\widehat{\xi_k} - \kappa\left(\int_{\mathcal{H}} u d\xi - 1\right). \end{aligned} \tag{63}$$

The first order optimality conditions are that the integral (63) is stationary with respect to infinitesimal perturbations in u and the boundary, as well as with respect to infinitesimal perturbations in the Lagrange multiplier functions λ, μ, and the number κ. We let $\dot{u}$ represent an infinitesimal perturbation in u, and similarly for other quantities. It is easier to calculate these conditions after transforming (63) into a form where u and u_{ξ_k} are factored. Specifically, we take (63) and apply integration by parts twice to the $\int_{\mathcal{H}} \lambda u_{\xi_i\xi_j} d\xi$ integral and then apply the chain rule (46) and integration by parts with the $d_{k,i}$ operator on the resulting B_k and Γ_k integrals, which, after some algebra, yields the form

$$\begin{aligned} \mathcal{L} = {} & \int_{\mathcal{H}} \left(-A_{ij}\lambda_{\xi_i\xi_j} + \frac{f_{zz}}{2}\xi_i\sigma_{ik}\sigma_{jk}\xi_j - \kappa\right) u\, d\xi \\ & + A_{ij}\int_{B_k} \left(\left[\begin{array}{c} -\varepsilon f_z b_k (\phi_k)_{\xi_i} (\phi_k)_{\xi_j} + \lambda_{\xi_i} (\phi_k)_{\xi_j} \\ + d_{k,i}\left((2\mu+\lambda)(\phi_k)_{\xi_j}\right) - \mu d_{k,i}\left((\phi_k)_{\xi_j}\right) \end{array} \right] u \right. \\ & \qquad \left. + \left[(\mu+\lambda)(\phi_k)_{\xi_i} (\phi_k)_{\xi_j}\right] u_{\xi_k} \right) d\widehat{\xi_k} \end{aligned}$$

$$
\begin{aligned}
&- A_{ij} \int_{\Gamma_k} \left(\left[\begin{array}{c} \varepsilon f_z c_k \left(\psi_k\right)_{\xi_i} \left(\psi_k\right)_{\xi_j} + \lambda_{\xi_i} \left(\psi_k\right)_{\xi_j} \\ +d_{k,i} \left((2\mu + \lambda) \left(\psi_k\right)_{\xi_j} \right) - \mu d_{k,i} \left(\left(\psi_k\right)_{\xi_j} \right) \end{array} \right] u \right. \\
&\qquad\qquad \left. + \left[(\mu + \lambda) \left(\psi_k\right)_{\xi_i} \left(\psi_k\right)_{\xi_j} \right] u_{\xi_k} \right) d\widehat{\xi}_k \\
&+ \kappa + I^{n-2}, \qquad (64)
\end{aligned}
$$

where, as in section 3.1, I^{n-2} represents the $(n-2)$-dimensional integrals produced from the integration by parts on the boundaries of the sets B_k and Γ_k.

The first order optimality condition with respect to perturbations in u is that for every $\dot{u}$ the variation in $\mathcal{L}$ should vanish:

$$
\begin{aligned}
0 = &\int_{\mathcal{H}} \left(-A_{ij} \lambda_{\xi_i \xi_j} + \frac{f_{zz}}{2} \xi_i \sigma_{ik} \sigma_{jk} \xi_j - \kappa \right) \dot{u} d\xi \\
&+ A_{ij} \int_{B_k} \left(\left[\begin{array}{c} -\varepsilon f_z b_k \left(\phi_k\right)_{\xi_i} \left(\phi_k\right)_{\xi_j} + \lambda_{\xi_i} \left(\phi_k\right)_{\xi_j} \\ +d_{k,i} \left((2\mu + \lambda) \left(\phi_k\right)_{\xi_j} \right) - \mu d_{k,i} \left(\left(\phi_k\right)_{\xi_j} \right) \end{array} \right] \dot{u} \right. \\
&\qquad\qquad \left. + \left[(\mu + \lambda) \left(\phi_k\right)_{\xi_i} \left(\phi_k\right)_{\xi_j} \right] \dot{u}_{\xi_k} \right) d\widehat{\xi}_k \\
&- A_{ij} \int_{\Gamma_k} \left(\left[\begin{array}{c} \varepsilon f_z c_k \left(\psi_k\right)_{\xi_i} \left(\psi_k\right)_{\xi_j} + \lambda_{\xi_i} \left(\psi_k\right)_{\xi_j} \\ +d_{k,i} \left((2\mu + \lambda) \left(\psi_k\right)_{\xi_j} \right) - \mu d_{k,i} \left(\left(\psi_k\right)_{\xi_j} \right) \end{array} \right] \dot{u} \right. \\
&\qquad\qquad \left. + \left[(\mu + \lambda) \left(\psi_k\right)_{\xi_i} \left(\psi_k\right)_{\xi_j} \right] \dot{u}_{\xi_k} \right) d\widehat{\xi}_k. \qquad (65)
\end{aligned}
$$

Here, we do not consider perturbing the probability at the (assumed optimal) intersections of the $2n$ buying and selling boundaries, thereby removing the effect of I^{n-2}. Since $\dot{u}$ is otherwise arbitrary, we get from the first line of (65) the following PDE within $\mathcal{H}$:

$$
0 = -A_{ij} \lambda_{\xi_i \xi_j} + \frac{f_{zz}}{2} \xi_i \sigma_{ik} \sigma_{jk} \xi_j - \kappa. \qquad (66)
$$

Since the matrix A_{ij} is positive definite, the third and fifth lines of (65) reduce to

$$
\mu + \lambda = 0 \qquad (67)
$$

on $\partial\mathcal{H}$. Applying (67) to the second and fourth lines of (65) along with the product rule, the chain rule (46), and A_{ij} being positive definite yields Neumann conditions for λ:

$$
\begin{aligned}
\lambda_{\xi_k} &= -\varepsilon b_k f_z \qquad \text{on } \beta_k, \\
\lambda_{\xi_k} &= \varepsilon c_k f_z \qquad \text{on } \gamma_k. \qquad (68)
\end{aligned}
$$

Next we look at the first order optimality condition with respect to perturbing the boundary. Infinitesimally perturbing the boundary function $\widehat{\beta}_k\left(\widehat{\xi}_k\right)$ causes a corresponding perturbation in $\phi_k(\xi)$, which we call $\dot{\phi}_k$ or, equivalently, $\dot{\widehat{\beta}}_k$. The definition of ϕ_k in (45) implies that $\left(\dot{\phi}_k\right)_{\xi_k} = 0$. We consider similar perturbations in $\psi_k(\xi)$ later. Since $\dot{\phi}_k = \dot{\widehat{\beta}}_k$, we have by the chain rule that, for example,

$\dot{\lambda}\left(\hat{\xi}_k, \hat{\beta}_k\left(\hat{\xi}_k\right)\right) = \lambda_{\xi_k}\dot{\hat{\beta}}_k = \lambda_{\xi_k}\dot{\phi}_k$, and so, recalling that $A_{ij} = A_{ji}$, the first order condition from perturbing the boundary functions takes the form

$$0 \;=\; A_{ij}\int_{B_k} I_{ijk}(\widehat{\xi}_k)\, d\widehat{\xi}_k,$$

where

$$I_{ijk} = \begin{bmatrix} -\varepsilon f_z b_k\left(2\left(\phi_k\right)_{\xi_i}\left(\dot{\phi}_k\right)_{\xi_j}\right) + \lambda_{\xi_i\xi_k}\dot{\phi}_k\left(\phi_k\right)_{\xi_j} + \lambda_{\xi_i}\left(\dot{\phi}_k\right)_{\xi_j} \\ +d_{k,i}\left[\lambda_{\xi_k}\dot{\phi}_k\left(\phi_k\right)_{\xi_j} + (2\mu+\lambda)\left(\dot{\phi}_k\right)_{\xi_j}\right] - \mu d_{k,i}\left(\left(\dot{\phi}_k\right)_{\xi_j}\right) \end{bmatrix} u$$

$$+\left[\lambda_{\xi_k}\dot{\phi}_k\left(\phi_k\right)_{\xi_i}\left(\phi_k\right)_{\xi_j} + (\mu+\lambda)\left(2\left(\phi_k\right)_{\xi_i}\left(\dot{\phi}_k\right)_{\xi_j}\right)\right] u_{\xi_k}. \tag{69}$$

(We do not perturb the boundary at the assumed optimal intersections of the buying and selling boundaries.) Since $\dot{\phi}_k$ has no ξ_k dependence, we have from the chain rule (46) that $d_{k,i}\dot{\phi}_k = \left(\dot{\phi}_k\right)_{\xi_i}$. Applying this fact, (67), (68), the chain rule (46), the product rule, integration by parts with the $d_{k,i}$ operator, and a significant amount of algebra allows us to reduce (69) to

$$0 = A_{ij}\int_{B_k} \varepsilon f_z b_k\left[u_{\xi_i}\left(\phi_k\right)_{\xi_j} + d_{k,i}\left(u\left(\phi_k\right)_{\xi_j}\right)\right]\dot{\phi}_k$$

$$+\left(d_{k,i}\left(\lambda_{\xi_k}\right) - \lambda_{\xi_k\xi_k}\left(\phi_k\right)_{\xi_i}\right)\left(\phi_k\right)_{\xi_j}\dot{\phi}_k u\, d\widehat{\xi}_k. \tag{70}$$

By the Neumann condition (48), the first line of (70) is zero, and by (68), $d_{k,i}\left(\lambda_{\xi_k}\right)$ in the second line is zero. Since the perturbation $\dot{\phi}_k$ is arbitrary, we have that $0 = A_{ij}\left(\phi_k\right)_{\xi_i}\left(\phi_k\right)_{\xi_j} u\lambda_{\xi_k\xi_k}$ on β_k. By the same logic, perturbing $\psi_k(\xi)$ leads to $0 = A_{ij}\left(\psi_k\right)_{\xi_i}\left(\psi_k\right)_{\xi_j} u\lambda_{\xi_k\xi_k}$ on γ_k. Since $u > 0$ and A_{ij} is positive definite, the final form of the first order condition from perturbing the boundary functions is just the second derivative boundary condition

$$\lambda_{\xi_k\xi_k} = 0 \qquad \text{on } \beta_k \text{ or } \gamma_k. \tag{71}$$

If we identify $\lambda(\xi)$ with

$$-\varepsilon^{\frac{4}{3}} f_4\left(z, \varepsilon^{-\frac{1}{3}}\xi, t\right),$$

we see that our PDE, (66), first derivative boundary conditions, (68), and second derivative boundary conditions, (71), completely match section 3.2's PDE, (60), first derivative boundary conditions, (56), and second derivative boundary conditions, (57). Therefore, the asymptotic analysis from section 3.2 justifies the assumptions in section 3.1 that were made to determine (36), our characterization of the optimization problem in terms of the opportunity and trading costs, and also (23), the single stock version of (36) obtained in section 2.2.

4. Density blow-up in the uncorrelated two stock case. Recall from the end of section 3.2 that, for two uncorrelated stocks, the hold region will be a rectangular region in ξ-space. We show in this section that in this case the density function, u, generally becomes singular at two opposite corners of this rectangle. This singular

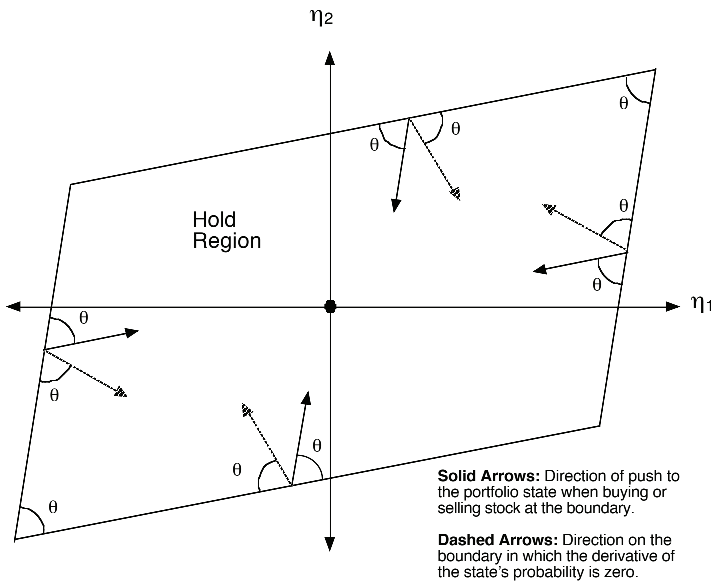

FIG. 2. *The hold region for two uncorrelated stocks in the η coordinate system.*

behavior is in marked contrast to the single stock case in section 2.2 where u was shown to be uniform. It appears to be a feature of the problem, not an artifact of the approximation strategy.

We start with the forward equation (43), where $A_{ij} = \frac{1}{2} a_{ik} a_{jk}$ and, from (34),

$$a_{ij} = m_i \delta_{ij} \sigma_j - (m_i)_z \, m_j \sigma_j \tag{72}$$

since $\sigma_{ij} = \sigma_i \delta_{ij}$ for uncorrelated stocks. Applying the transformation

$$\eta_i = a_{ij}^{-1} \xi_j,$$

the forward equation becomes the Laplace equation

$$0 = u_{\eta_i \eta_i}.$$

Since the matrix a_{ij} in (72) is not, in general, diagonal, the rectangular hold region, $\mathcal{H}$, in ξ-space will be transformed, in general, into a *nonrectangular* parallelogram, $\mathcal{H}^\eta$, in η-space (see Figure 2).

When transformed into η-space, the oblique boundary conditions (48) take the form

$$\begin{aligned} 0 &= -a_{ki} \left[2u_{\eta_i} - a_{jk}^{-1} a_{ki} u_{\eta_j} \right] \qquad \text{on } B_k^\eta, \\ 0 &= a_{ki} \left[2u_{\eta_i} - a_{jk}^{-1} a_{ki} u_{\eta_j} \right] \qquad \text{on } \Gamma_k^\eta, \end{aligned}$$

where we do not sum over k. A long, but straightforward, calculation reconfirms the following known geometric fact: On the boundary, in η-space, the derivative of u is zero in a direction that forms an angle, θ, to the boundary, where θ is the acute angle formed by the parallelogram's sides. We also note that if this direction where the derivative of u equals zero is flipped about the normal to the boundary, it corresponds to the direction of push due to stock sales or purchases at the boundary. Again, see Figure 2.

From well-known results for Laplace's equation and oblique boundary conditions on polygonal domains, we have that u is constant only when the parallelogram in η-space is a rectangle, that is, only when $\theta = \frac{\pi}{2}$. For any other angle, $\theta \neq \frac{\pi}{2}$, the density becomes infinite (but remains integrable, of course) as the two obtuse vertices of the parallelogram are approached, while the density approaches zero as the two acute vertices are approached. At a simple financial level, this can be understood as the vectors for purchasing and selling stock at the boundary pushing the state of the portfolio into the obtuse corners and away from the acute corners. The general theory for singularities in corners can be found in Grisvard [16]. The discussion in Trefethen and Williams [17] specializes to the present application.

For a better understanding, along with determining the asymptotic rate at which the density becomes infinite or goes to zero, we magnify our view of the vertex so that near the vertex we can approximate the problem as acting on an infinite wedge whose corner is the vertex. A solution in polar coordinates to Laplace's equation on the wedge is

$$u(r, \phi) = Cr^m \cos\left(m(\phi + \phi_1)\right)$$

for constants C, m, and ϕ_1. The oblique boundary conditions yield, for some constant ϕ_2, that $\cos(\theta - m\phi_2) = \cos(\theta + m(\theta + \phi_2)) = 0$, which implies that $2\theta + m\theta = n\pi$ for some integer n. When $\theta = \frac{\pi}{2}$, there is no push into the vertices from buying or selling, so it follows that u is constant, and therefore $m = 0$. Inserting this fact into $2\theta + m\theta = n\pi$ yields that $n = 1$, and therefore we have that m, the exponent for the rate of blow-up or disappearance of u at the vertex, is

$$m = \frac{\pi}{\theta} - 2.$$

For $\theta < \frac{\pi}{2}$, we have that $m > 0$, so this gives the rate at which u decays to zero as we approach the acute vertices. For $\frac{\pi}{2} < \theta < \pi$, we have that $-1 < m < 0$, which gives the rate of blow-up for u at the obtuse vertices. Note that since $-1 < m$, we always have that the density remains integrable near the vertex.

Appendix. Derivation of the second derivative "smooth pasting" boundary condition. In this appendix we use calculus of variations to show how the first variation of the boundary leads to a second derivative boundary condition.

Let $O[\cdot]$ be a given differential operator so that $f_t + O[f] = 0$ is a PDE for $f(x,t)$ subject to the maximum principle, as is the case for the parabolic equations encountered in this paper. Let $x \in \Omega(t) \subset \mathbf{R}^n$, where $\partial\Omega(t)$ is piecewise smooth, $\Omega(t)$ is continuous in $t \in [0,T]$, and, most importantly, $\Omega(t)$ is defined so that $\Omega^T = \bigcup_{t\in[0,T]} (\Omega(t), t)$ is the region in $\mathbf{R}^n \times [0,T]$ that optimizes the solution, $f(x,t)$, to our PDE

$$f_t + O[f] = 0 \qquad \text{for } (x,t) \in \Omega^T \tag{73}$$

subject to a given initial condition and the boundary condition

$$v(x,t) \cdot \nabla_x f(x,t) = 0 \qquad \text{for } x \in \partial\Omega(t). \tag{74}$$

Now consider a parameterized mapping $M_\varepsilon : \Omega(t) \to \Omega_\varepsilon(t)$ from the optimal region to suboptimal regions when $\varepsilon \neq 0$ defined by

$$x \mapsto x + \varepsilon w(x,t), \tag{75}$$

where $w(x,t) \in \mathbf{R}^n$ is a fixed, but arbitrary, vector field subject to $w(x,0) = 0$ so that the initial condition is unchanged by ε. On $\Omega_\varepsilon^T = \bigcup_{t\in[0,T]} (\Omega_\varepsilon(t), t)$ we have that the suboptimal solutions, $f(x,t,\varepsilon)$, still satisfy the PDE (73), the initial condition, and the boundary condition

$$v(x,t) \cdot \nabla_x f(x,t,\varepsilon) = 0 \qquad \text{for } x \in \partial\Omega_\varepsilon(t).$$

Taking the derivative of the boundary condition with respect to ε at locations where $\partial\Omega_\varepsilon(t)$ is smooth, we obtain

$$\nabla_x \left(v(x,t) \cdot \nabla_x f(x,t,\varepsilon)\right) \cdot \frac{\partial x}{\partial \varepsilon} + v(x,t) \cdot \nabla_x \frac{\partial f}{\partial \varepsilon} = 0 \qquad \text{for } x \in \partial\Omega_\varepsilon. \tag{76}$$

Noting from (75) that ε affects x on the boundary through the relation $x(\varepsilon) = x + \varepsilon w(x,t)$, we have that $\frac{\partial x}{\partial \varepsilon} = w(x,t)$, and since f is optimized at $\varepsilon = 0$, from the maximum principle, we have that $\frac{\partial f}{\partial \varepsilon}\big|_{\varepsilon=0} = 0$; therefore, at $\varepsilon = 0$, (76) reduces to

$$\nabla_x \left(v(x,t) \cdot \nabla_x f(x,t,0)\right) \cdot w(x,t) = 0 \qquad \text{for } x \in \partial\Omega_0. \tag{77}$$

Since w is arbitrary and the optimal region $\Omega = \Omega_0$, we see from (77) that the optimality of Ω has generated the second derivative boundary condition

$$\nabla_x \left(v(x,t) \cdot \nabla_x f(x,t)\right) = 0 \qquad \text{for } x \in \partial\Omega. \tag{78}$$

Note, however, that the first derivative condition (74) already implies that the directional derivative of $v(x,t) \cdot \nabla_x f(x,t)$ is zero in any direction tangent to $\partial\Omega$, so the only new implication of (78) is that the directional derivative of $v(x,t) \cdot \nabla_x f(x,t)$ is zero in any direction that is *transverse* to the boundary. This means that while (78) contains n conditions, only one condition is not immediately implied by the first derivative condition (74).

REFERENCES

[1] R. C. Merton, *Continuous-Time Finance*, revised ed., Blackwell, Oxford, UK, 1992.

[2] J. J. P. Magill and G. M. Constantinides, *Portfolio selection with transaction costs*, J. Econom. Theory, 13 (1976), pp. 245–263.

[3] J. Céa, *Conception optimale ou identification de formes: Calcul rapide de la dérivée directionnelle de la fonction coût* [Optimal conception or identification of forms: Rapid calculation of the directional derivative of the cost function], RAIRO Modél. Math. Anal. Numér., 20 (1986), pp. 371–402.

[4] J. Goodman and D. N. Ostrov, *An Option to Reduce Transaction Costs*, preprint.

[5] C. Atkinson and S. Mokkhavesa, *Multi-asset portfolio optimization with transaction cost*, Appl. Math. Finance, 11 (2004), pp. 329–364.

[6] S. E. Shreve and H. M. Soner, *Optimal investment and consumption with transaction costs*, Ann. Appl. Probab., 4 (1994), pp. 609–692.

[7] K. Janeček and S. E. Shreve, *Asymptotic analysis for optimal investment and consumption with transaction costs*, Finance Stoch., 8 (2004), pp. 181–206.

[8] M. H. A. Davis, V. G. Panas, and T. Zariphopoulou, *European option pricing with transaction costs*, SIAM J. Control Optim., 31 (1993), pp. 470–493.

[9] A. E. Whalley and P. Wilmott, *An asymptotic analysis of an optimal hedging model for option pricing with transaction costs*, Math. Finance, 7 (1997), pp. 307–324.

[10] K. Muthuraman and S. Kumar, *Multidimensional portfolio optimization with proportional transaction costs*, Math. Finance, 16 (2006), pp. 301–335.

[11] C. Atkinson and P. Ingpochai, *The influence of correlation on multi-asset portfolio optimization with transaction costs*, J. Comput. Finance, 10 (2006), pp. 53–96.

[12] S. Mokkhavesa and C. Atkinson, *Perturbation solution of optimal portfolio theory with transaction costs for any utility function*, IMA J. Manag. Math., 13 (2002), pp. 131–151.

[13] J. Goodman and D. N. Ostrov, *Balancing Small Transaction Costs with Loss of Optimal Allocation in Single and Multiple Stock Portfolios*, working paper, 2007; available online at http://www.math.nyu.edu/financial_mathematics/content/02_financial/Working_Paper_Series.html.
[14] A. Dixit, *A simplified treatment of the theory of optimal regulation of Brownian motion*, J. Econom. Dynam. Control, 15 (1991), pp. 657–673.
[15] B. Dumas, *Super contact and related optimality conditions*, J. Econom. Dynam. Control, 15 (1991), pp. 675–685.
[16] P. Grisvard, *Boundary Value Problems in Non-Smooth Domains*, Pitman, London, 1985.
[17] L. N. Trefethen and R. J. Williams, *Conformal mapping solution of Laplace's equation on a polygon with oblique derivative boundary conditions*, J. Comput. Appl. Math., 14 (1986), pp. 227–249.