Nonuniqueness for the Vanishing Viscosity Solution with Fixed Initial Condition in a Nonstrictly Hyperbolic System of Conservation Laws

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Abstract

We consider the Riemann problem for a system of two decoupled, nonstrictly hyperbolic, Burgers-like conservation equations with added artificial viscosity. We analytically establish two different vanishing viscosity limits for the solution of this system, which correspond to the two cases where one of the viscosities vanishes much faster than the other. This is done without altering the initial condition as is necessary with travelling wave methods. Numerical evidence is then provided to show that when the two viscosities vanish at the same rate, the solution converges to a limit that lies strictly between the two previously established limits. Finally, we use control theory to explain the mechanism behind this nonuniqueness behavior, which indicates other systems of nonstrictly hyperbolic conservation laws where nonuniqueness will occur.

1 Introduction

In this paper we establish nonuniqueness for the vanishing viscosity limit of a system of two nonstrictly hyperbolic conservation laws without using travelling wave methods. Our analysis will primarily involve the Hamilton-Jacobi form

$$\begin{aligned} \mathbf{u}_t + \mathbf{H}(\mathbf{u}_x) &= 0 \\ \mathbf{u}(x,0) &= \mathbf{g}(x) \end{aligned} \qquad (1)$$

where the domain of the solution is $(x,t) \in \Pi^+ = \mathbf{R} \times \mathbf{R}^+$, the solution $\mathbf{u} : \Pi^+ \to \mathbf{R}^2$ is continuous though not necessarily differentiable, the initial condition $\mathbf{g} : \mathbf{R} \to \mathbf{R}^2$ is Lipschitz continuous, and the Hamiltonian $\mathbf{H} : \mathbf{R}^2 \to \mathbf{R}^2$ is smooth. The Hamilton-Jacobi form can easily be transformed into conservation law form by differentiating (1) with respect to x and defining $\mathbf{f} = \mathbf{H}, \mathbf{U} = \mathbf{u}_x$ where $\mathbf{u}$ is differentiable, and $\mathbf{G} = \mathbf{g}_x$ where $\mathbf{g}$ is differentiable, which yields:

$$\begin{aligned} \mathbf{U}_t + \mathbf{f}(\mathbf{U})_x &= \mathbf{0} \\ \mathbf{U}(x,0) &= \mathbf{G}(x). \end{aligned} \qquad (2)$$

Similarly, integration transforms the conservation law form (2) back into Hamilton-Jacobi form (1) .

The ability to define a unique solution to (2), and therefore to (1), depends upon the nature of the flux, $\mathbf{f} = \begin{bmatrix} f_1 & f_2 \end{bmatrix}^T$, and the size of the initial condition, $\mathbf{G}$. If the eigenvalues, $\lambda_1(\mathbf{U})$ and $\lambda_2(\mathbf{U})$, to the Jacobian matrix

$$D\mathbf{f}(\mathbf{U}) = \begin{bmatrix} \frac{\partial f_1}{\partial U_1} & \frac{\partial f_1}{\partial U_2} \\ \frac{\partial f_2}{\partial U_1} & \frac{\partial f_2}{\partial U_2} \end{bmatrix}$$

are real and distinct for all $\mathbf{U} = \begin{bmatrix} U_1 & U_2 \end{bmatrix}^T$, then the system (2) is strictly hyperbolic, and we can look to define the solution via two methods, both of which are motivated by physics.

The first method is the entropy solution method. Generally, a solution $\mathbf{U}$ of (2) will fail to be continuous — corresponding to a solution $\mathbf{u}$ to (1) failing to be differentiable — along curves in the (x,t) plane, which are called shock curves. Let the function ξ be defined so that $(\xi(t), t)$ represents a generic shock curve. The shock curve is called a Lax entropy shock (see [1]) if, for almost all t and for $i = 1$ or 2 (but not for both $i = 1$ and 2),

$$\lim_{x \nearrow \xi(t)} \lambda_i(\mathbf{U}(x,t)) > \xi'(t) > \lim_{x \searrow \xi(t)} \lambda_i(\mathbf{U}(x,t)). \qquad (3)$$

For $i = 1$ or 2, an i^{th} family's characteristic curve, $x_i(t)$, satisfies the ODE

$$\frac{dx_i}{dt} = \lambda_i(\mathbf{U}(x,t))$$

in regions of Π^+ where $\mathbf{U}$ is continuous. So, geometrically, (3) and strict hyperbolicity imply that characteristic curves are not generated at Lax entropy shocks. That is, if we have the value of i where (3) holds, then the i^{th} family's characteristics terminate on (that is, enter but do not exit) the shock, and, for the other value of i, the i^{th} family's characteristics pass through the Lax shock. Specifically, no i^{th} family's characteristics can be generated at (that is, exit but do not enter) a Lax entropy shock. An entropy solution is a solution where all the shocks are Lax shocks. There is a unique entropy solution to (2) if the total variation of the initial condition, $\mathbf{G}$, is sufficiently small and, for each $i = 1$ or 2, we have either linear degeneracy

$$[D\lambda_i(\mathbf{U})]^T [r_i(\mathbf{U})] = 0 \qquad \text{for all } \mathbf{U}$$

or genuine nonlinearity

$$[D\lambda_i(\mathbf{U})]^T [r_i(\mathbf{U})] \neq 0 \qquad \text{for all } \mathbf{U}$$

where $r_i(\mathbf{U})$ is the right eigenvector of $D\mathbf{f}(\mathbf{U})$ corresponding to the eigenvalue λ_i and $D\lambda_i = \begin{bmatrix} \frac{\partial \lambda_i}{\partial U_1} & \frac{\partial \lambda_i}{\partial U_2} \end{bmatrix}^T$ (see [2]).

The second method is the vanishing viscosity method. Here we would like to define $\mathbf{U}$, the solution to (2) , by $\lim_{\varepsilon \to 0} \mathbf{U}^\varepsilon(x,t)$ where $\mathbf{U}^\varepsilon$ is the unique smooth solution to the "viscous" form of the equation :

$$\begin{aligned} \mathbf{U}_t^\varepsilon + \mathbf{F}(\mathbf{U}^\varepsilon)_x &= \varepsilon \mathbf{D} \mathbf{U}_{xx}^\varepsilon \\ \mathbf{U}^\varepsilon(x,0) &= \mathbf{G}(x) \end{aligned} \tag{4}$$

where $\varepsilon > 0$ is called the viscosity and $\mathbf{D}$, which is a positive definite constant matrix, is called the viscosity matrix. If the total variation of $\mathbf{G}$ is small and $\mathbf{D}$ is the identity matrix, then it is known (see [3]) that $\mathbf{U}$ can be defined by this limit. Further, when the linear degeneracy/genuine nonlinearity conditions above are met, the vanishing viscosity solution agrees with the entropy solution. Work by Majda and Pego [4] using the flux in (4) linearized about a fixed state, $\mathbf{U}_0^\varepsilon$, suggests that $\mathbf{D}$ can only affect the stability of the solution. That is, the vanishing viscosity limit may not exist for some $\mathbf{D}$, but when it does exist, it should be the same regardless of $\mathbf{D}$.

The nonstrictly hyperbolic case differs significantly from the hyperbolic case. The system (2) is called nonstrictly hyperbolic when λ_1 and λ_2, the eigenvalues of $D\mathbf{f}(\mathbf{U})$, are real, but not necessarily distinct[1]. That is, there is a nonempty subset, Σ, of $\mathbf{R}^2$ where $\lambda_1(\mathbf{U}) = \lambda_2(\mathbf{U})$ for $\mathbf{U} \in \Sigma$. Much of what is known about the nonstrictly hyperbolic case has come from lines of research that began in the mid 1980's, and, as is the case in this article, the focus of this research has been on the Riemann problem; that is, a solution of (2) with an initial condition, $\mathbf{G}(x)$, of the form

$$\mathbf{G}(x) = \begin{cases} \mathbf{U}_L & \text{for } x < 0 \\ \mathbf{U}_R & \text{for } x > 0 \end{cases}$$

where $\mathbf{U}_L$ and $\mathbf{U}_R$ are constant vectors. For example, in [8] and [9], Keyfitz and Kranzer looked at the Riemann problem for (2) within a specific class of fluxes where Σ is a curve in $\mathbf{R}^2$. For their specific class, they found a unique solution by removing the strictness of the inequalities in the Lax condition (3) . Soon after that, Schaeffer, Schearer et al. [7], [10] considered a specific class of cubic fluxes where Σ is an isolated point. They divided the class into four groups and found that obtaining existence in the fourth group required allowing for the presence of "undercompressive waves" (also called "transitional waves"). These waves are shocks for which the strict inequalities in (3) are reversed and, therefore, characteristics are generated at the shock. Since the notion of an "entropy solution" had to be abandoned, attention was focused instead on the vanishing viscosity method to recover a unique solution. In particular, attention was focused on the travelling wave method as a way of indicating the vanishing viscosity solution.

In the travelling wave method, a solution to the Riemann problem for (2) is admissible if all of its shocks satisfy the viscous profile criterion, which we next quickly explain (but see, for example, chapter 8 of [5] for more detail). For the Riemann problem shocks have constant speed and the limits of the solution from the two sides of any shock are constants. For a generic shock, define s to be the shock speed, $\xi'(t)$, and define $\mathbf{U}_-$ and $\mathbf{U}_+$ to be the constant limits of the solution as the shock is approached from the left and the right, respectively. If we assume near the shock that the solution to the viscous form of the equation, (4) , can be expressed as a travelling wave — that is,

[1]Often hyperbolicity is defined to also require two linearly independent eigenvectors (see for example [5] or [6]), however, in the line of papers discussed here, specifically in [7], this is not part of the definition.

$\mathbf{U}^{\varepsilon}(x,t) = \mathbf{V}(\xi)$ where $\xi = \frac{x-st}{\varepsilon}$ — then the PDE in (4) for $\mathbf{U}^{\varepsilon}$ corresponds to the following ODE for $\mathbf{V}$:

$$\mathbf{V}' = \mathbf{D}^{-1}\left[-s(\mathbf{V} - \mathbf{U}_-) + \mathbf{f}(\mathbf{V}) - \mathbf{f}(\mathbf{U}_-)\right].$$

A shock satisfies the viscous profile criterion if there is an orbit for this ODE from the point $\mathbf{U}_-$, which is clearly a critical point of the ODE, to $\mathbf{U}_+$, which is also a critical point by the Rankine-Hugoniot condition.

The travelling wave method yields admissible undercompressive waves; it can also be used to show that, unlike the strictly hyperbolic case, $\mathbf{D}$ can lead to nonuniqueness. Specifically, different vanishing viscosity solutions may exist for different values of $\mathbf{D}$ (see, for example, [11] or [12]). However, for $\varepsilon \neq 0$, the travelling wave method requires altering the Riemann initial conditions near the origin in a different way for each shock so that the solution near the shock can be expressed as a function of ξ. It is only in the limit as $\varepsilon \to 0$ that we recover the two constant states of the Riemann initial data. Further, Freistühler, Liu, Azevedo, Marchesin, Plohr, and Zumbrun (see [13], [14], [15], [16], [17], and [18]) have shown cases of extreme sensitivity of the solution, even in the asymptotic limit as $t \to \infty$, to arbitrarily small in L^1alterations near $x = 0$ in the Riemann initial condition.

In this paper we will look at the vanishing viscosity limit without using methods even remotely connected to travelling waves. Our method has the advantage of permitting the Riemann initial conditions to stay constant as the viscosity vanishes, although it appears not to be extendable to the wide class of systems covered by the travelling wave method. We will apply our method to a simple nonstrictly hyperbolic example system in Hamilton-Jacobi form to recover both the presence of undercompressive waves and the nonuniqueness of the solution (in the sense that the vanishing viscosity limit depends upon the manner in which the viscosity vanishes).

The conservation form of our example Riemann system is the following decoupled system which contains quadratic fluxes:

$$\begin{aligned} U_t + U_x^2 &= \varepsilon_1 U_{xx} \quad \text{where } U(x,0) = \begin{cases} a \text{ if } x \leq 0 \\ -a \text{ if } x > 0 \end{cases} \\ V_t + V_x^2 - U_x^2 &= \varepsilon_2 V_{xx} \quad \text{where } V(x,0) = 0 \end{aligned} \tag{5}$$

where a is any fixed positive constant. Since the total variation of the initial condition is $2a$ and a can be arbitrarily small, we will see that nonuniqueness can occur for arbitrarily small total variation of the initial condition. Note that we have replaced $\varepsilon\mathbf{D}$ in (2) with positive viscosities ε_1 and ε_2. Since $\mathbf{f} = \begin{bmatrix} U^2 \\ V^2 - U^2 \end{bmatrix}$, we have that $D\mathbf{f}(\mathbf{U}) = \begin{bmatrix} 2U & 0 \\ -2U & 2V \end{bmatrix}$ so the two eigenvalues of $D\mathbf{f}$, $2U$ and $2V$, are always real, and since these eigenvalues are the same if $U = V$, system (5) is nonstrictly hyperbolic. Note that Σ is a line, not an isolated point, so we do not fit into any of the four groups for quadratic fluxes classified by Issacson et al. in [19].

As indicated above, in this paper we will work with the Hamilton-Jacobi form of (5), which is

$$\begin{aligned} u_t + (u_x)^2 &= \varepsilon_1 u_{xx} \qquad \text{where } u(x,0) = -a\,|x| \\ v_t + (v_x)^2 - (u_x)^2 &= \varepsilon_2 v_{xx} \qquad \text{where } v(x,0) = 0. \end{aligned} \tag{6}$$

In section 2 we analyze the first equation in (6), the decoupled equation for u, which is Burgers' equation. It is known that for the given initial condition, u_x will converge to the travelling wave solution $-a\tanh\left(\frac{ax}{\varepsilon_1}\right)$. In this section we establish bounds for the speed of this convergence that will be sufficient for our needs in section 3 where we substitute our expression for u_x from section 2 into the second equation in (6), the equation for v. Our analysis will establish that when we let ε_1 vanish much faster than ε_2 vanishes, we get one limit for v at $x = 0$, whereas when we let ε_2 vanish much faster than ε_1 vanishes, we get a different limit for v at $x = 0$. In section 4, we give numerical evidence indicating a third (intermediate) limit for v at $x = 0$ when ε_1 and ε_2 vanish at the same rate. In section 5, we explain this nonuniqueness behavior from a control theory perspective, which will not only provide some intuition for why the vanishing viscosity solution is not unique, but will also allow us to quickly compute the vanishing viscosity limits for v in the cases considered in sections 3 and 4 when $x \neq 0$. The control theory perspective will also indicate a larger class of decoupled systems to which (6) belongs where this nonuniqueness behavior occurs.

2 Analysis of the independent equation for u

The independent equation for u is Burgers' equation

$$u_t + (u_x)^2 = \varepsilon_1 u_{xx}, \tag{7}$$

which we will analyze on the infinite strip $x \in (-\infty, \infty)$, $t \in [0, T]$ in the (x,t) plane, where T is an arbitrary positive number. The initial condition is

$$u(x,0) = -a\,|x| \qquad \text{where } a > 0. \tag{8}$$

Our goal is to show that $u_x \approx -a\tanh\left(\frac{ax}{\varepsilon_1}\right)$ or, more specifically, that $(u_x)^2$, which we will insert into the equation for v in the next section, equals $a^2\tanh^2\left(\frac{ax}{\varepsilon_1}\right) + \Delta^{\varepsilon_1}(x,t)$, where the error term, $\Delta^{\varepsilon_1}(x,t)$, will not affect the value of v once $\varepsilon_1 \to 0$ (regardless of how we let $\varepsilon_2 \to 0$).

Our analysis of (7) and (8) starts with the classical Hopf-Cole transformation for Burgers' equation. We substitute $u = -\varepsilon_1 \ln(w)$, which transforms (7) and (8) into the linear heat equation

$$\begin{aligned} w_t &= \varepsilon_1 w_{xx} \\ w(x,0) &= e^{\frac{a|x|}{\varepsilon_1}} \end{aligned}$$

whose solution is well known:

$$w(x,t) = \frac{1}{\sqrt{4\pi\varepsilon_1 t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4\varepsilon_1 t}} e^{\frac{a|y|}{\varepsilon_1}}\,dy.$$

Retransforming back to u and differentiating with respect to x gives

$$u_x(x,t) = \frac{x}{2t} - \frac{1}{2t}\frac{\int_{-\infty}^{\infty} y e^{-\frac{(x-y)^2}{4\varepsilon_1 t}} e^{\frac{a|y|}{\varepsilon_1}}\,dy}{\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4\varepsilon_1 t}} e^{\frac{a|y|}{\varepsilon_1}}\,dy}. \tag{9}$$

Next, because of the $|y|$ in (9), we split both of the integrals in (9) into two integrals: one over positive y, the other over negative y. This allows us to use standard techniques to evaluate all four integrals (completing the square, etc.), which, after some calculus and a lot of algebra including finally recombining the positive y and negative y integral results, yields

$$\int_{-\infty}^{\infty} y e^{-\frac{(x-y)^2}{4\varepsilon_1 t}} e^{\frac{a|y|}{\varepsilon_1}}\,dy = \sqrt{\pi\varepsilon_1 t}\, e^{\frac{a^2 t}{\varepsilon_1}} \left[\begin{array}{c} e^{\frac{ax}{\varepsilon_1}}(2at+x)\left(1+\operatorname{erf}\left(\frac{2at+x}{\sqrt{4\varepsilon_1 t}}\right)\right) \\ -e^{\frac{-ax}{\varepsilon_1}}(2at-x)\left(1+\operatorname{erf}\left(\frac{2at-x}{\sqrt{4\varepsilon_1 t}}\right)\right) \end{array} \right]$$

and

$$\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4\varepsilon_1 t}} e^{\frac{a|y|}{\varepsilon_1}}\,dy = \sqrt{\pi\varepsilon_1 t}\, e^{\frac{a^2 t}{\varepsilon_1}} \left[\begin{array}{c} e^{\frac{ax}{\varepsilon_1}}\left(1+\operatorname{erf}\left(\frac{2at+x}{\sqrt{4\varepsilon_1 t}}\right)\right) \\ +e^{\frac{-ax}{\varepsilon_1}}\left(1+\operatorname{erf}\left(\frac{2at-x}{\sqrt{4\varepsilon_1 t}}\right)\right) \end{array} \right].$$

Substituting these into (9) yields

$$u_x(x,t) = -a\frac{e^{\frac{ax}{\varepsilon_1}}\left(1+\operatorname{erf}\left(\frac{2at+x}{\sqrt{4\varepsilon_1 t}}\right)\right) - e^{\frac{-ax}{\varepsilon_1}}\left(1+\operatorname{erf}\left(\frac{2at-x}{\sqrt{4\varepsilon_1 t}}\right)\right)}{e^{\frac{ax}{\varepsilon_1}}\left(1+\operatorname{erf}\left(\frac{2at+x}{\sqrt{4\varepsilon_1 t}}\right)\right) + e^{\frac{-ax}{\varepsilon_1}}\left(1+\operatorname{erf}\left(\frac{2at-x}{\sqrt{4\varepsilon_1 t}}\right)\right)}. \tag{10}$$

Now we divide the domain $x \in (-\infty, \infty)$, $t \in [0, T]$ into four regions, where each region has properties that make the analysis of our expression for u_x in (10) easier. We define region 1 to be the region where $\frac{\varepsilon_1^{\frac{1}{4}}}{a} < t \le T$, $2at - x \ge \varepsilon_1^{\frac{1}{4}}$, and $2at + x \ge \varepsilon_1^{\frac{1}{4}}$. This region has a trapezoidal shape in the (x,t) plane which is symmetric about the t-axis and is where the most important behavior for u_x occurs. Region 2 is the small time region $0 \le t \le \frac{\varepsilon_1^{\frac{1}{4}}}{a}$. Region 3 is where $\frac{\varepsilon_1^{\frac{1}{4}}}{a} < t \le T$ and $2at - x < \varepsilon_1^{\frac{1}{4}}$, and region 4, which is the reflection of region 3 about the t-axis, is where $\frac{\varepsilon_1^{\frac{1}{4}}}{a} < t \le T$ and $2at + x < \varepsilon_1^{\frac{1}{4}}$.

For region 1, we first consider the half of the region where $x \ge 0$. Dividing the expression for u_x in (10) by $\left(1+\operatorname{erf}\left(\frac{2at+x}{\sqrt{4\varepsilon_1 t}}\right)\right)$ yields

$$u_x(x,t) = -a\frac{e^{\frac{ax}{\varepsilon_1}} - e^{\frac{-ax}{\varepsilon_1}}\frac{\left(1+\operatorname{erf}\left(\frac{2at-x}{\sqrt{4\varepsilon_1 t}}\right)\right)}{\left(1+\operatorname{erf}\left(\frac{2at+x}{\sqrt{4\varepsilon_1 t}}\right)\right)}}{e^{\frac{ax}{\varepsilon_1}} + e^{\frac{-ax}{\varepsilon_1}}\frac{\left(1+\operatorname{erf}\left(\frac{2at-x}{\sqrt{4\varepsilon_1 t}}\right)\right)}{\left(1+\operatorname{erf}\left(\frac{2at+x}{\sqrt{4\varepsilon_1 t}}\right)\right)}}. \tag{11}$$

Since the error function has the asymptotic formula

$$\operatorname{erf}(x) \equiv \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt \sim \begin{cases} 1 - \frac{1}{\sqrt{\pi}x}e^{-x^2} & \text{for } x \text{ large and positive} \\ -1 - \frac{1}{\sqrt{\pi}x}e^{-x^2} & \text{for } x \text{ large and negative,} \end{cases}$$

the fact that $2at - x \ge \varepsilon_1^{\frac{1}{4}}$ and $2at + x \ge \varepsilon_1^{\frac{1}{4}}$ in this region imply that as ε_1 gets small, we obtain

$$\frac{\left(1+\operatorname{erf}\left(\frac{2at-x}{\sqrt{4\varepsilon_1 t}}\right)\right)}{\left(1+\operatorname{erf}\left(\frac{2at+x}{\sqrt{4\varepsilon_1 t}}\right)\right)} = 1 + O\left(\varepsilon_1^{\frac{1}{4}}e^{-\frac{1}{4T\sqrt{\varepsilon_1}}}\right). \tag{12}$$

Also, since $x > 0$, we have that $e^{-\frac{ax}{\varepsilon_1}} \le 1$, therefore, from (11) and (12), we see that

$$u_x(x,t) = -a\tanh\left(\frac{ax}{\varepsilon_1}\right) + O\left(\varepsilon_1^{\frac{1}{4}}e^{-\frac{1}{4T\sqrt{\varepsilon_1}}}\right). \tag{13}$$

On the other hand, for the half of region 1 where $x < 0$, we divide (10) by $\left(1 + \operatorname{erf}\left(\frac{2at-x}{\sqrt{4\varepsilon_1 t}}\right)\right)$and apply the same arguments as above noting that now $e^{\frac{ax}{\varepsilon_1}} \leq 1$, which leads to the same result so (13) holds for all (x, t) in region 1.

Region 2, the small time region $0 \leq t \leq \frac{\varepsilon_1^{\frac{1}{4}}}{a}$, is difficult to analyze carefully near the origin. Fortunately, because the time interval is chosen so small in this region, we will only need the crude bounds

$$-a \leq u_x \leq a \tag{14}$$

which follow from noting that the fraction in (10) must be bounded between -1 and 1.

In region 1, the properties of the error function were key; in regions 3 and 4, the properties of the exponential functions are key. For region 3, we first rewrite (10) in the form

$$u_x(x,t) = -a \frac{1 - e^{\frac{-2ax}{\varepsilon_1}} \frac{\left(1+\operatorname{erf}\left(\frac{2at-x}{\sqrt{4\varepsilon_1 t}}\right)\right)}{\left(1+\operatorname{erf}\left(\frac{2at+x}{\sqrt{4\varepsilon_1 t}}\right)\right)}}{1 + e^{\frac{-2ax}{\varepsilon_1}} \frac{\left(1+\operatorname{erf}\left(\frac{2at-x}{\sqrt{4\varepsilon_1 t}}\right)\right)}{\left(1+\operatorname{erf}\left(\frac{2at+x}{\sqrt{4\varepsilon_1 t}}\right)\right)}}.$$

Our two restrictions, $\frac{\varepsilon_1^{\frac{1}{4}}}{a} < t \leq T$ and $2at - x < \varepsilon_1^{\frac{1}{4}}$, imply that $x > \varepsilon_1^{\frac{1}{4}}$ and $2at + x > 3\varepsilon_1^{\frac{1}{4}}$ so $\operatorname{erf}\left(\frac{2at+x}{\sqrt{4\varepsilon_1 t}}\right) \geq 0$ and therefore,

$$u_x(x,t) = -a \frac{1 + O\left(e^{-2a\varepsilon_1^{-\frac{3}{4}}}\right)}{1 + O\left(e^{-2a\varepsilon_1^{-\frac{3}{4}}}\right)} = -a + O\left(e^{-2a\varepsilon_1^{-\frac{3}{4}}}\right). \tag{15}$$

Similarly, $\tanh\left(\frac{ax}{\varepsilon_1}\right) = \frac{e^{\frac{ax}{\varepsilon_1}} - e^{\frac{-ax}{\varepsilon_1}}}{e^{\frac{ax}{\varepsilon_1}} + e^{\frac{-ax}{\varepsilon_1}}} = \frac{1-e^{\frac{-2ax}{\varepsilon_1}}}{1+e^{\frac{-2ax}{\varepsilon_1}}} = 1 + O\left(e^{-2a\varepsilon_1^{-\frac{3}{4}}}\right)$, which, when combined with (15) , yields

$$u_x(x,t) = -a \tanh\left(\frac{ax}{\varepsilon_1}\right) + O\left(e^{-2a\varepsilon_1^{-\frac{3}{4}}}\right). \tag{16}$$

The analysis for region 4 parallels the analysis for region 3. We now rewrite (10) as

$$u_x(x,t) = -a\frac{e^{\frac{2ax}{\varepsilon_1}}\frac{\left(1+\operatorname{erf}\left(\frac{2at+x}{\sqrt{4\varepsilon_1 t}}\right)\right)}{\left(1+\operatorname{erf}\left(\frac{2at-x}{\sqrt{4\varepsilon_1 t}}\right)\right)}-1}{e^{\frac{2ax}{\varepsilon_1}}\frac{\left(1+\operatorname{erf}\left(\frac{2at+x}{\sqrt{4\varepsilon_1 t}}\right)\right)}{\left(1+\operatorname{erf}\left(\frac{2at-x}{\sqrt{4\varepsilon_1 t}}\right)\right)}+1},$$

and then use the fact that the region's restrictions imply that $x < -\varepsilon_1^{\frac{1}{4}}$ and $2at - x > 3\varepsilon_1^{\frac{1}{4}}$ so $\operatorname{erf}\left(\frac{2at-x}{\sqrt{4\varepsilon_1 t}}\right) \geq 0$ and again we obtain the expression in (16).

We now square the expressions for u_x in (13), (14), and (16) and use the fact that $|\tanh(z)| \leq 1$ to obtain our desired expression for $(u_x)^2$:

$$[u_x(x,t)]^2 = a^2\tanh^2\left(\frac{ax}{\varepsilon_1}\right) + \Delta^{\varepsilon_1}(x,t) \tag{17}$$

where, for $\frac{\varepsilon_1^{\frac{1}{4}}}{a} < t \leq T$, $\Delta^{\varepsilon_1}(x,t)$ is $O\left(\varepsilon_1^{\frac{1}{4}}e^{-\frac{1}{4T\sqrt{\varepsilon_1}}}\right)$ uniformly in x and t as $\varepsilon_1 \to 0$ and, for $0 \leq t \leq \frac{\varepsilon_1^{\frac{1}{4}}}{a}$, $|\Delta^{\varepsilon_1}(x,t)| \leq a^2$.

3 Analysis of the v equation

We are now ready to look at the v equation

$$\begin{aligned} v_t + (v_x)^2 - (u_x)^2 &= \varepsilon_2 v_{xx} \\ v(x,0) &= 0 \end{aligned}$$

where $(u_x)^2$ is given by (17). The analysis for v where $x \neq 0$ is relatively straightforward and will be contained in section 5. Our analysis in this section will soon center on the more difficult (and interesting) situation where $x = 0$. We will show that we get different vanishing viscosity limits for v depending upon how we let $(\varepsilon_1, \varepsilon_2)$ approach $(0,0)$.

We begin by applying the Hopf-Cole transformation, $v = -\varepsilon_2 \ln w$, just as we did in the previous section, which again yields a linear PDE; specifically,

$$\begin{aligned} w_t + \frac{(u_x)^2}{\varepsilon_2}w &= \varepsilon_2 w_{xx} \\ w(x,0) &= 1. \end{aligned}$$

The solution to this PDE can be expressed via the Feynman-Kac stochastic formula

$$w(x,t) = E\left[w(X(t),0)e^{-\frac{1}{\varepsilon_2}\int_0^t [u_x(X(s),t-s)]^2 ds}\right]$$

subject to the dynamics

$$X(0) = x \quad \text{and} \quad dX = \sqrt{2\varepsilon_2}dB$$

where B is a Brownian motion.

Applying the initial condition, $w(x,0) = 1$, and then retransforming from w back to v gives

$$\begin{aligned} v(x,t) &= -\varepsilon_2 \ln\left(E\left[e^{-\frac{1}{\varepsilon_2}\int_0^t [u_x(X(s),t-s)]^2 ds}\right]\right) \\ \text{where } X(0) &= x \quad \text{and} \quad dX = \sqrt{2\varepsilon_2}dB. \end{aligned}$$

Now we substitute (17), our expression for $(u_x)^2$ from the previous section, to obtain

$$v(x,t) = -\varepsilon_2 \ln\left(E\left[e^{-\frac{1}{\varepsilon_2}\int_0^t a^2 \tanh^2\left(\frac{aX(s)}{\varepsilon_1}\right) + \Delta^{\varepsilon_1}(X(s),t-s)ds}\right]\right).$$

From our knowledge of the bounds on the behavior of the error term Δ^{ε_1} in (17) and the fact that $t \leq T$, we have that

$$\begin{aligned} & \int_0^t \Delta^{\varepsilon_1}(X(s),t-s)ds \qquad (18) \\ &= \int_0^{t-\frac{\varepsilon_1^{\frac{1}{4}}}{a}} \Delta^{\varepsilon_1}(X(s),t-s)ds + \int_{t-\frac{\varepsilon_1^{\frac{1}{4}}}{a}}^t \Delta^{\varepsilon_1}(X(s),t-s)ds \\ &= O\left(\varepsilon_1^{\frac{1}{4}} e^{-\frac{1}{4T\sqrt{\varepsilon_1}}}\right) + O\left(\varepsilon_1^{\frac{1}{4}}\right) = O\left(\varepsilon_1^{\frac{1}{4}}\right). \end{aligned}$$

We note that this $O\left(\varepsilon_1^{\frac{1}{4}}\right)$ bound is independent of the stochastic behavior; therefore, (18) becomes

$$v(x,t) = -\varepsilon_2 \ln\left(E\left[e^{-\frac{1}{\varepsilon_2}\int_0^t a^2 \tanh^2\left(\frac{aX(s)}{\varepsilon_1}\right)ds}\right]\right) + O\left(\varepsilon_1^{\frac{1}{4}}\right). \qquad (19)$$

Next we concentrate on the stochastic expression in (19) :

$$E\left[e^{-\frac{1}{\varepsilon_2}\int_0^t a^2 \tanh^2\left(\frac{aX(s)}{\varepsilon_1}\right)ds}\right] \quad \text{where } X(0) = x \quad \text{and} \quad dX = \sqrt{2\varepsilon_2}dB. \qquad (20)$$

The expected value in (20) can be bounded above and below by using Jensen's inequality to exploit the convexity of the exponential function. The upper bound on the expected value obtained from Jensen's inequality is

$$E\left[e^{-\frac{1}{\varepsilon_2}\int_0^t a^2\tanh^2\left(\frac{aX(s)}{\varepsilon_1}\right)ds}\right] \le \frac{1}{t}E\left[\int_0^t e^{-\frac{t}{\varepsilon_2}a^2\tanh^2\left(\frac{aX(s)}{\varepsilon_1}\right)}ds\right], \qquad (21)$$

whereas the lower bound obtained is

$$E\left[e^{-\frac{1}{\varepsilon_2}\int_0^t a^2\tanh^2\left(\frac{aX(s)}{\varepsilon_1}\right)ds}\right] \ge e^{-\frac{1}{\varepsilon_2}E\left[\int_0^t a^2\tanh^2\left(\frac{aX(s)}{\varepsilon_1}\right)ds\right]}. \qquad (22)$$

From the definition of Brownian motion we know the stochastic dynamics in (20) imply that $p(\xi, s)$, the probability density function for the location, ξ, of X at any given time $s \in [0, t]$, is

$$p(\xi, s) = \frac{1}{\sqrt{4\pi\varepsilon_2 s}}e^{-\frac{(\xi-x)^2}{4\varepsilon_2 s}}. \qquad (23)$$

Both the bounds (21) and (22) can be interpreted as extremes of stochastic behavior that conform to this probability density function. The right hand side of (21) corresponds to dynamics where the paths, $X(s)$, keep the same value as much as possible while still conforming to $p(\xi, s)$ in (23) . The right hand side of (22) corresponds to dynamics where the paths, $X(s)$, randomize the values they take as much as possible while still conforming to $p(\xi, s)$ in (23) .

Inserting the bounds (21) and (22) into (19) and applying (23) gives us an upper bound on $v(x, t)$

$$v(x,t) \le \int_{-\infty}^{\infty}\int_0^t a^2\tanh^2\left(\frac{a\xi}{\varepsilon_1}\right)\frac{1}{\sqrt{4\pi\varepsilon_2 s}}e^{-\frac{(\xi-x)^2}{4\varepsilon_2 s}}ds d\xi + O\left(\varepsilon_1^{\frac{1}{4}}\right) \qquad (24)$$

and a lower bound on $v(x, t)$

$$v(x,t) \ge -\varepsilon_2 \ln\left[\frac{1}{t}\int_{-\infty}^{\infty}\int_0^t e^{-\frac{a^2 t}{\varepsilon_2}\tanh^2\left(\frac{a\xi}{\varepsilon_1}\right)}\frac{1}{\sqrt{4\pi\varepsilon_2 s}}e^{-\frac{(\xi-x)^2}{4\varepsilon_2 s}}ds d\xi\right] + O\left(\varepsilon_1^{\frac{1}{4}}\right). \qquad (25)$$

We now analyze the integrals in these two bounds more carefully for the case where $x = 0$. Note that for the rest of this section and for the entire next section we restrict our analysis to $v(0, t)$, and we return to the easier to analyze case where $x \neq 0$ in Section 5.

For the integral in the upper bound (24) with $x = 0$, we apply Fubini's theorem to change the order of integration and then make the substitution $\xi = z\sqrt{4\varepsilon_2 s}$, which gives us

$$\int_{-\infty}^{\infty}\int_0^t a^2 \tanh^2\left(\frac{a\xi}{\varepsilon_1}\right)\frac{1}{\sqrt{4\pi\varepsilon_2 s}}e^{-\frac{\xi^2}{4\varepsilon_2 s}}dsd\xi \tag{26}$$

$$= \; a^2\int_0^t\int_{-\infty}^{\infty}\tanh^2\left(\frac{2a\sqrt{\varepsilon_2 s}}{\varepsilon_1}z\right)\sqrt{\frac{1}{\pi}}e^{-z^2}dzds.$$

Since we are ready to look at the behavior of $v(0,t)$ as $\underline{\varepsilon} = (\varepsilon_1, \varepsilon_2)$ approaches $(0,0)$, we will now denote v by $v^{\underline{\varepsilon}}$ to emphasize the $\underline{\varepsilon}$ dependence. A rough upper bound on $v^{\underline{\varepsilon}}(0,t)$ is easy to obtain by inserting (26) into (24) and taking the limit as the viscosities vanish:

$$\begin{aligned}
&\limsup_{(\varepsilon_1,\varepsilon_2)\to(0,0)} v^{\underline{\varepsilon}}(0,t) \qquad\qquad (27)\\
\leq\; &\limsup_{(\varepsilon_1,\varepsilon_2)\to(0,0)}\left[a^2\int_0^t\int_{-\infty}^{\infty}\tanh^2\left(\frac{2a\sqrt{\varepsilon_2 s}}{\varepsilon_1}z\right)\sqrt{\frac{1}{\pi}}e^{-z^2}dzds + O\left(\varepsilon_1^{\frac{1}{4}}\right)\right]\\
\leq\; &\limsup_{(\varepsilon_1,\varepsilon_2)\to(0,0)}\left[a^2\int_0^t\int_{-\infty}^{\infty}\sqrt{\frac{1}{\pi}}e^{-z^2}dzds + O\left(\varepsilon_1^{\frac{1}{4}}\right)\right]\\
=\; &\limsup_{(\varepsilon_1,\varepsilon_2)\to(0,0)}\left[a^2 t + O\left(\varepsilon_1^{\frac{1}{4}}\right)\right]\\
=\; &a^2 t.
\end{aligned}$$

What is more interesting is to let the viscosities vanish along any path where the ratio $\frac{\varepsilon_2}{(\varepsilon_1)^2}$ goes to 0 as $(\varepsilon_1, \varepsilon_2)$ goes to $(0,0)$.[2] Specifically, we parameterize ε_1 and ε_2 by $\rho \in \mathbf{R}^+$ so that not only does $\lim_{\rho\to 0}\varepsilon_1(\rho) = 0$ and $\lim_{\rho\to 0}\varepsilon_2(\rho) = 0$ but also $\lim_{\rho\to 0}\frac{\varepsilon_2(\rho)}{(\varepsilon_1(\rho))^2} = 0$, and then we can apply dominated convergence to the first line in (27) to get a much tighter upper bound:

$$\begin{aligned}
&\limsup_{\rho\to 0} v^{\underline{\varepsilon}(\rho)}(0,t) \qquad\qquad (28)\\
\leq\; &\limsup_{\rho\to 0}\left[a^2\int_0^t\int_{-\infty}^{\infty}\tanh^2\left(\frac{2a\sqrt{\varepsilon_2(\rho)s}}{\varepsilon_1(\rho)}z\right)\sqrt{\frac{1}{\pi}}e^{-z^2}dzds + O\left(\varepsilon_1^{\frac{1}{4}}\right)\right]
\end{aligned}$$

[2]The travelling wave method restricts viscosities to vanish along straight lines through the origin in the (ξ_1, ξ_2) plane; that is, ξ_1 and ξ_2 are restricted to be constant multiples of each other. Note that we are not encumbered by that restriction here.

$$
\begin{aligned}
&= \left[a^2 \int_0^t \int_{-\infty}^{\infty} \tanh^2\left(\limsup_{\rho\to 0}\left(\frac{2a\sqrt{\varepsilon_2(\rho)s}}{\varepsilon_1(\rho)}z\right)\right)\sqrt{\frac{1}{\pi}}e^{-z^2}dzds + \limsup_{\rho\to 0} O\left(\varepsilon_1^{\frac{1}{4}}(\rho)\right)\right] \\
&= 0.
\end{aligned}
$$

Now we examine the lower bound (25) . For the integral term in (25) with $x = 0$, we also apply Fubini's theorem to change the order of integration, but make the slightly different substitution $\xi = z\sqrt{4s}$, which gives us

$$
\begin{aligned}
&-\varepsilon_2 \ln\left[\frac{1}{t}\int_{-\infty}^{\infty}\int_0^t e^{-\frac{a^2t}{\varepsilon_2}\tanh^2\left(\frac{a\xi}{\varepsilon_1}\right)}\frac{1}{\sqrt{4\pi\varepsilon_2 s}}e^{-\frac{\xi^2}{4\varepsilon_2 s}}dsd\xi\right] \qquad (29) \\
&= -\varepsilon_2 \ln\left[\frac{1}{t}\frac{1}{\sqrt{\pi\varepsilon_2}}\int_0^t\int_{-\infty}^{\infty} e^{-\left[\frac{a^2t}{\varepsilon_2}\tanh^2\left(\frac{a\sqrt{4s}}{\varepsilon_1}z\right)+\frac{z^2}{\varepsilon_2}\right]}dzds\right].
\end{aligned}
$$

We note that (29) can be used to establish a rough lower bound that is analogous to the rough upper bound in (27) :

$$
\begin{aligned}
&\liminf_{(\varepsilon_1,\varepsilon_2)\to(0,0)} v^{\underline{\varepsilon}}(0,t) \qquad (30) \\
&\geq \liminf_{(\varepsilon_1,\varepsilon_2)\to(0,0)}\left\{-\varepsilon_2\ln\left[\frac{1}{t}\frac{1}{\sqrt{\pi\varepsilon_2}}\int_0^t\int_{-\infty}^{\infty}e^{-\left[\frac{a^2t}{\varepsilon_2}\tanh^2\left(\frac{a\sqrt{4s}}{\varepsilon_1}z\right)+\frac{z^2}{\varepsilon_2}\right]}dzds\right]+O\left(\varepsilon_1^{\frac{1}{4}}\right)\right\} \\
&\geq \liminf_{(\varepsilon_1,\varepsilon_2)\to(0,0)}\left\{-\varepsilon_2\ln\left[\frac{1}{t}\frac{1}{\sqrt{\pi\varepsilon_2}}\int_0^t\int_{-\infty}^{\infty}e^{-\frac{z^2}{\varepsilon_2}}dzds\right]+O\left(\varepsilon_1^{\frac{1}{4}}\right)\right\} \\
&= \liminf_{(\varepsilon_1,\varepsilon_2)\to(0,0)}\left\{-\varepsilon_2\ln\left(\frac{t\sqrt{\pi\varepsilon_2}}{t\sqrt{\pi\varepsilon_2}}\right)+O\left(\varepsilon_1^{\frac{1}{4}}\right)\right\} \\
&= 0
\end{aligned}
$$

Note that (28) and (30) establish, for example, that $\lim_{\varepsilon_1\to 0}\lim_{\varepsilon_2\to 0} v^{\underline{\varepsilon}}(0,t) = 0$.

Now we consider tighter lower bounds that result from choosing specific paths by which the viscosites can vanish. If we are only interested in reversing the order of our limits from before — that is determining $\lim_{\varepsilon_2\to 0}\lim_{\varepsilon_1\to 0} v^{\underline{\varepsilon}}(0,t)$ — we need to compute $\lim_{\varepsilon_2\to 0}\lim_{\varepsilon_1\to 0}$ of (29), which can be accomplished quickly using dominated convergence and the definition of the L^∞ norm:

$$
\begin{aligned}
&\lim_{\varepsilon_2\to 0}\lim_{\varepsilon_1\to 0} -\varepsilon_2\ln\left[\frac{1}{t}\frac{1}{\sqrt{\pi\varepsilon_2}}\int_0^t\int_{-\infty}^{\infty}e^{-\left[\frac{a^2t}{\varepsilon_2}\tanh^2\left(\frac{a\sqrt{4s}}{\varepsilon_1}z\right)+\frac{z^2}{\varepsilon_2}\right]}dzds\right] \\
&= \lim_{\varepsilon_2\to 0}\lim_{\varepsilon_1\to 0}\varepsilon_2\frac{\ln(t^2\pi\varepsilon_2)}{2} - \lim_{\varepsilon_2\to 0}\lim_{\varepsilon_1\to 0}\ln\left[\int_0^t\int_{-\infty}^{\infty}e^{-\left[a^2t\tanh^2\left(\frac{a\sqrt{4s}}{\varepsilon_1}z\right)+z^2\right]\frac{1}{\varepsilon_2}}dzds\right]^{\varepsilon_2}
\end{aligned}
$$

$$
\begin{aligned}
&= -\lim_{\varepsilon_2\to 0}\ln\left[\int_0^t\int_{-\infty}^{\infty} e^{-\left[a^2t+z^2\right]\frac{1}{\varepsilon_2}}dzds\right]^{\varepsilon_2}\\
&= -\ln\left[\left\|e^{-\left(a^2t+z^2\right)}\right\|_{L^\infty}\right]\\
&= -\ln e^{-a^2t}\\
&= a^2t.
\end{aligned}
$$

Combining this with (25) and the rough bound (27) establish that $\lim_{\varepsilon_2\to 0}\lim_{\varepsilon_1\to 0} v^{\underline{\varepsilon}}(0,t) = a^2t$, so we see that switching the order of the limits yields different solutions.

As with the upper bound analysis, however, we also want to obtain the limit a^2t for a larger set of vanishing viscosities. Specifically, we now parameterize ε_1 and ε_2 by $\rho \in \mathbf{R}^+$ so that not only does $\lim_{\rho\to 0}\varepsilon_1(\rho) = 0$ and $\lim_{\rho\to 0}\varepsilon_2(\rho) = 0$ but also

$$
\lim_{\rho\to 0}\varepsilon_1(\rho)e^{\frac{a^2t}{\varepsilon_2(\rho)}} \le M \tag{31}
$$

for some finite number M. Noting that for any $\varepsilon_2 > 0$

$$
\tanh^2(z) \ge \begin{cases} 0 & \text{if } |z| < \varepsilon_2^{-\frac{1}{4}} \\ \tanh^2\left(\varepsilon_2^{-\frac{1}{4}}\right) & \text{if } |z| \ge \varepsilon_2^{-\frac{1}{4}} \end{cases}
$$

we have for our expression in (29) that

$$
\begin{aligned}
&-\varepsilon_2\ln\left[\frac{1}{t}\frac{1}{\sqrt{\pi\varepsilon_2}}\int_0^t\int_{-\infty}^{\infty} e^{-\left[\frac{a^2t}{\varepsilon_2}\tanh^2\left(\frac{a\sqrt{4s}}{\varepsilon_1}z\right)+\frac{z^2}{\varepsilon_2}\right]}dzds\right] \qquad (32)\\
\ge\; &-\varepsilon_2\ln\left[\frac{2}{\sqrt{t^2\pi\varepsilon_2}}\left[e^{-\frac{a^2t}{\varepsilon_2}\tanh^2\left(\varepsilon_2^{-\frac{1}{4}}\right)}\int_0^t\int_{\frac{\varepsilon_2^{-\frac{1}{4}}\varepsilon_1}{2a\sqrt{s}}}^{\infty} e^{-\frac{z^2}{\varepsilon_2}}dzds + \int_0^t\int_0^{\frac{\varepsilon_2^{-\frac{1}{4}}\varepsilon_1}{2a\sqrt{s}}} e^{-\frac{z^2}{\varepsilon_2}}dzds\right]\right]\\
\ge\; &-\varepsilon_2\ln\left[\frac{2}{\sqrt{t^2\pi\varepsilon_2}}\left[e^{-\frac{a^2t}{\varepsilon_2}\tanh^2\left(\varepsilon_2^{-\frac{1}{4}}\right)}\int_0^t\int_0^{\infty} e^{-\frac{z^2}{\varepsilon_2}}dzds + \int_0^t\int_0^{\frac{\varepsilon_2^{-\frac{1}{4}}\varepsilon_1}{2a\sqrt{s}}} dzds\right]\right]\\
=\; &-\varepsilon_2\ln\left[e^{-\frac{a^2t}{\varepsilon_2}\tanh^2\left(\varepsilon_2^{-\frac{1}{4}}\right)} + \frac{2\varepsilon_2^{-\frac{3}{4}}\varepsilon_1}{a\sqrt{t\pi}}\right]\\
=\; &a^2t\tanh^2\left(\varepsilon_2^{-\frac{1}{4}}\right) - \varepsilon_2\ln\left[1+\frac{2\varepsilon_2^{-\frac{3}{4}}\varepsilon_1}{a\sqrt{t\pi}}e^{\frac{a^2t}{\varepsilon_2}\tanh^2\left(\varepsilon_2^{-\frac{1}{4}}\right)}\right]
\end{aligned}
$$

$$
\begin{aligned}
&\geq \; a^2 t \tanh^2\left(\varepsilon_2^{-\frac{1}{4}}\right) - \varepsilon_2 \ln\left[1 + \frac{2\varepsilon_2^{-\frac{3}{4}}\varepsilon_1}{a\sqrt{t\pi}} e^{\frac{a^2 t}{\varepsilon_2}}\right] \\
&\geq \; a^2 t \tanh^2\left(\varepsilon_2^{-\frac{1}{4}}\right) - \varepsilon_2^{\frac{1}{4}} \frac{2\varepsilon_1}{a\sqrt{t\pi}} e^{\frac{a^2 t}{\varepsilon_2}}
\end{aligned}
$$

where the last inequality follows from $-\ln(1+x) \geq -x$. Inserting this last result, along with (29), into (25) and applying our restriction (31) yields the lower bound

$$
\begin{aligned}
&\liminf_{\rho \to 0} v^{\underline{\varepsilon}(\rho)}(0,t) \qquad (33) \\
\geq \; &\liminf_{\rho \to 0} \left[a^2 t \tanh^2\left(\varepsilon_2^{-\frac{1}{4}}(\rho)\right) - \varepsilon_2^{\frac{1}{4}}(\rho) \frac{2\varepsilon_1(\rho)}{a\sqrt{t\pi}} e^{\frac{a^2 t}{\varepsilon_2(\rho)}} + O\left(\varepsilon_1^{\frac{1}{4}}(\rho)\right)\right] \\
= \; &a^2 t.
\end{aligned}
$$

Combining our upper limit bounds (27) and (28) with our lower limit bounds (30) and (33) , we have that the vanishing viscosity limit for $v^{\underline{\varepsilon}(\rho)}(0,t)$ subject to the restriction that $\lim_{\rho \to 0} \frac{\varepsilon_2(\rho)}{(\varepsilon_1(\rho))^2} = 0$ is

$$
\lim_{\rho \to 0} v^{\underline{\varepsilon}(\rho)}(0,t) = 0,
$$

whereas the vanishing viscosity limit for $v^{\underline{\varepsilon}(\rho)}(0,t)$ subject to the restriction that $\lim_{\rho \to 0} \varepsilon_1(\rho) e^{\frac{a^2 t}{\varepsilon_2(\rho)}} \leq M$ for some finite M is

$$
\lim_{\rho \to 0} v^{\underline{\varepsilon}(\rho)}(0,t) = a^2 t.
$$

Loosely speaking, if ε_2 stays much smaller than ε_1 as they both shrink to zero, then the vanishing viscosity limit for $v(0,t) = 0$, but if ε_1 stays much smaller than ε_2 as they both shrink to zero, then the vanishing viscosity limit for $v(0,t) = a^2 t$.

4 Numerical Simulations for v when $\varepsilon_1 = \varepsilon_2 \to 0$

Now that we have established that the vanishing viscosity solution for $v(0,t)$ is different when ε_2 stays much smaller than ε_1 as they both shrink to zero versus when ε_1 stays much smaller than ε_2 as they both shrink to zero, it is natural to wonder what happens in the case where ε_1 and ε_2 shrink to zero at exactly the same rate. Does the vanishing viscosity solution for $v(0,t)$ when $\varepsilon_1 = \varepsilon_2$ correspond to either of our previous extreme cases — 0 or $a^2 t$ — or to a limit strictly between these two cases? Our numerical results in this section will suggest that the vanishing viscosity solution when $\varepsilon_1 = \varepsilon_2$ is strictly between the two extreme cases; specifically, it is approximately equal to $0.62a^2 t$.

The fact that $\varepsilon_1 = \varepsilon_2$ allows us to use scaling to simplify our calculations: by defining $\varepsilon \equiv \varepsilon_1 = \varepsilon_2$, we can also define

$$\begin{aligned} \tilde{x} &= \frac{x}{\varepsilon} \\ \tilde{t} &= \frac{t}{\varepsilon} \\ \tilde{u} &= \frac{u}{\varepsilon} \\ \tilde{v} &= \frac{v}{\varepsilon}, \end{aligned}$$

which scales (6) into

$$\begin{aligned} \tilde{u}_{\tilde{t}} + (\tilde{u}_{\tilde{x}})^2 &= \tilde{u}_{\tilde{x}\tilde{x}} \qquad \text{where } \tilde{u}(\tilde{x},0) = -a\,|\tilde{x}| \\ \tilde{v}_{\tilde{t}} + (\tilde{v}_{\tilde{x}})^2 - (\tilde{u}_{\tilde{x}})^2 &= \tilde{v}_{\tilde{x}\tilde{x}} \qquad \text{where } \tilde{v}(\tilde{x},0) = 0. \end{aligned} \tag{34}$$

When ε_1 and ε_2 vanish in the two extreme cases, the ratio $\frac{v(0,t)}{t}$ converges to 0 or to a^2, so we now want to look at what happens to $\frac{v(0,t)}{t} = \frac{\tilde{v}(0,\tilde{t})}{\tilde{t}}$ as $\varepsilon \to 0$ in the solution to (34) . Since, for any fixed t, $\varepsilon \to 0$ implies that $\tilde{t} \to \infty$, this means we want to numerically investigate $\lim\limits_{\tilde{t}\to\infty} \frac{\tilde{v}(0,\tilde{t})}{\tilde{t}}$ to see if it exists, and if it does, to see if it equals 0, a^2 or something in between.

At each time step, we first approximate $\tilde{u}$ using the decoupled first equation in (34) and then we simulate $\tilde{v}$ using the second equation in (34). For each equation, we apply Godunov's finite difference method adapted to the Hamilton-Jacobi form (see, for example, Leveque [20], for the Conservation Law form) to simulate the first order nonlinear effects in (34) , and we use the standard three point centered difference approximation to simulate the second order terms. Boundary conditions for $\tilde{u}$ are provided by noting that for $|\tilde{x}|$ sufficiently large, the second order term $\tilde{u}_{\tilde{x}\tilde{x}}$ in the first equation in

(34) has little effect, so we can approximate the solution for large $|\tilde{x}|$ by $\tilde{u}(\tilde{x},\tilde{t}) = -a|\tilde{x}| - a^2\tilde{t}$, the solution obtained if we set $\tilde{u}_{\tilde{x}\tilde{x}} = 0$. In practice, $|\tilde{x}| = 5$ is sufficiently large for this approximation to have no effect on the simulation. The boundary conditions for v are determined by 1) using the three points closest to the boundary to approximate $\tilde{v}_{\tilde{x}\tilde{x}}$, and 2) noting that Godunov's simulation of the nonlinear first derivative terms never requires the values of v at points outside the boundary. This is because in the non-viscous case, the characteristics for v generated at $t = 0$ from outside the region $|\tilde{x}| \leq k$ (for any positive constant k) never enter the region $|\tilde{x}| \leq k$ at a later time, which we show in the next section.

We ran three simulations up to $\tilde{t} = 24$ for different values of a and mesh sizes. In each case the ratio $r\left(\tilde{t}\right) = \frac{\tilde{v}(0,\tilde{t})}{\tilde{t}}$ monotonically increased and converged quite quickly, as is seen in the following chart:

a	$\Delta\tilde{x}$	$\Delta\tilde{t}$	$r(1)$	$r(6)$	$r(12)$	$r(24)$
1	0.01	2×10^{-5}	0.61245	0.61667	0.61708	0.61729
1	0.005	5×10^{-6}	0.61145	0.61673	0.61725	0.61751
3	0.01	2×10^{-5}	5.5427	5.5466	5.5469	5.5471.

The chart gives evidence that $r\left(\tilde{t}\right)$ converges to approximately $0.62a^2$, which implies that when $\varepsilon_1 = \varepsilon_2$, the vanishing viscosity limit at $x = 0$ is $v(0,t) \approx 0.62a^2t$. In the next section, we show how to construct $v(x,t)$ from $v(0,t)$ and also explain the nonuniqueness for $v(0,t)$ using control theory.

5 Control Theory Perspective

We can develop intuition about why the vanishing viscosity solution is not unique for our example (6) and many related decoupled systems by looking at these systems from a control theory perspective. This perspective will not only allow us to better understand the underlying principle behind our nonuniqueness phenomena in a more general setting, but will also allow us to complete our example by giving a formula for $v(x,t)$ through using $V(t) = v(0,t)$, which was determined in the previous two sections.

Consider the viscous system of equations

$$\begin{aligned} u_t + H_1(u_x) &= \varepsilon_1 u_{xx} \qquad u(x,0) = u_0(x) \\ v_t + H_2(u_x, v_x) &= \varepsilon_2 v_{xx} \qquad v(x,0) = v_0(x) \end{aligned} \tag{35}$$

where H_1, H_2, u_0, and v_0 are Lipschitz continuous and the v equation's Hamiltonian, $H_2(r,p)$, for any fixed r, is both convex in p and grows superlinearly in p (i.e., $\lim_{p\to\pm\infty} \frac{H_2(r,p)}{|p|} = \infty$). The convexity and superlinear growth imply that we can determine the Lagrangian, L, corresponding to this Hamiltonian using $L(r,-p) = \max_{a\in\mathbf{R}}[ap - H_2(r,a)]$, which, after solving the equation for u in (35) , allows us to express the control theory formulation of the solution to the v equation in (35); specifically,

$$v(x,t) = \min_{\alpha(\cdot,\cdot)\in M} E\left[\int_0^t L\left(u_x\left(y(s), t-s\right), \alpha\left(y(s),s\right)\right) ds + v_0(x(t))\right] \tag{36}$$

subject to the feedback control dynamics

$$dy(s) = \alpha\left(y(s),s\right) ds + \sqrt{2\varepsilon_2}dB, \qquad y(0) = x, \tag{37}$$

where B is a Brownian motion and M is the set of nonanticipating, measurable functions on the domain $\mathbf{R}\times[0,t]$. (See, for example, [21].) Note that ε_2 is a measure of the amount of stochastic noise in the path $y(s)$, so as ε_2 gets small, the path $y(s)$ becomes more controllable through the specific α function chosen.

Now consider the independent u equation

$$\begin{aligned} u_t + H_1(u_x) &= \varepsilon_1 u_{xx} \\ u(x,0) &= u_0(x). \end{aligned}$$

For any $\varepsilon_1 > 0$, u_x is a continuous function, but in the limit as $\varepsilon_1 \to 0$, u_x typically converges to a function where u_x is discontinuous along shock curves in the (x,t) plane. Assume there is a shock which can be described by $(\xi(t),t)$ for some function ξ over a domain which is a subset of $(0,T]$. Further, let us define $u_x^-(t) \equiv \lim_{x\nearrow\xi(t)} u_x(x,t)$ and $u_x^+(t) \equiv \lim_{x\searrow\xi(t)} u_x(x,t)$; that is, $u_x^-(t)$ is the limit of u_x from the left of the shock and $u_x^+(t)$ is the limit of u_x from the right. For $\varepsilon_1 > 0$, u_x gets smoothed out, so instead of a shock curve, we have a thin transition region where the value of u_x moves quickly, but continuously, from $u_x^-(t)$ to $u_x^+(t)$ and the thickness of this transition region goes to zero as ε_1 goes to zero.

Now assume there is a time interval of positive measure on which there exists some function $\hat{u}_x(t)$ that takes values strictly between $u_x^-(t)$ and $u_x^+(t)$ and where $L\left(\hat{u}_x(t), \xi'(t)\right) < L\left(u_x^\pm(t), \xi'(t)\right)$. In this case, the Lagrangian is smaller somewhere inside the transition region than just outside it, so there can be incentive for $y(s)$ to stay in the transition region if it is possible to do so. Whether this is possible is dictated by the relative sizes of ε_1 and ε_2 since ε_1 corresponds to the thickness of the transition region and ε_2 corresponds to the amount of stochastic noise in $y(s)$, and, therefore, the ability of $y(s)$ to stay within the transition region.

We next explain this phenomenon within the context of our example problem. For our example we have $H_2(u_x, v_x) = v_x^2 - u_x^2$, and $v_0(x) = 0$ in equation (35) , so $L(u_x, \alpha) = u_x^2 + \frac{\alpha^2}{4}$ and (36) becomes

$$v(x,t) = \min_{\alpha(\cdot,\cdot)\in M} E\left[\int_0^t u_x^2\left(y(s),s\right) + \frac{\alpha^2\left(y(s),s\right)}{4} ds\right] \tag{38}$$

subject to the dynamics in (37) . Since

$$u_x(x,t) = -a\tanh\left(\frac{ax}{\varepsilon_1}\right) + O\left(\varepsilon_1^{\frac{1}{4}}\right), \tag{39}$$

as $\varepsilon_1 \longrightarrow 0$, u_x converges to

$$u_x = \begin{cases} -a & \text{if } x > 0 \\ a & \text{if } x < 0 \end{cases}$$

so there is a shock on $x = 0$; that is, $\xi(t) = 0$, $u_x^-(t) = a$ and $u_x^+(t) = -a$. We now choose $\hat{u}_x(t) = 0$, which satisfies the two desired properties: $\hat{u}_x(t)$ is strictly between $u_x^-(t)$ and $u_x^+(t)$ and, since $u_x = 0$ minimizes the value of the Lagrangian, $L(\hat{u}_x(t), 0) < L(u_x^{\pm}(t), 0)$. We want to choose α functions that cause $y(s)$ to move to the shock and stay there (that is, stay at $x = 0$ where $u_x = 0$) if possible. The thickness of our transition region is $O(\varepsilon_1)$ as we can see by (39) , our expression for u_x, but if ε_2, the magnitude of the stochastic noise, is too much larger than ε_1, then we are unable to stay in the transition region for any measurable time, and we do not reduce the cost of the Lagrangian by choosing α functions that lead to the shock.

This is the behavior that we quantified in section 3. Specifically, if $\varepsilon_2 = o\left(\varepsilon_1^2\right)$ as $\varepsilon_1 \longrightarrow 0$, then the stochastic noise is so weak that we can stay near $x = 0$ where u_x is small. In fact, if we set $u_x = \hat{u}_x = 0$ and $\alpha\left(y(s),s\right) = 0$ in (38) — which corresponds to staying on $x = 0$ — then we have that $v(0,t) = 0$ as we proved in section 3. On the other hand, if $\varepsilon_1 = O\left(e^{-\frac{a^2 t}{\varepsilon_2}}\right)$ as $\varepsilon_2 \longrightarrow 0$, then the stochastic noise is severe and prevents paths where

u_x^2 is significantly less than a^2. In fact, if we set $u_x^2 = a^2$ in (38) , we see that choosing $\alpha\left(y(s),s\right) = 0$ minimizes the expected value and so we get $v(0,t) = a^2 t$ as was proved in section 3. If neither ε_1 nor ε_2 dominate the other as they shrink to zero, then it is unclear which effect is stronger: we may get $v(0,t) = 0$ or $v(0,t) = a^2 t$ or something strictly between these two limits — as was indicated in the previous section for $\varepsilon_1 = \varepsilon_2$ — or something undefined. (We know that the limit cannot be less than 0 or greater than $a^2 t$ by our rough bounds (27) and (30) .)

We are now ready to look at the vanishing viscous limit of $v(x,t)$ for $x \neq 0$. Because $(u_x)^2$ converges uniformly to a^2 as $\varepsilon_1 \to 0$ over any region $|x| \geq \delta$ where $\delta > 0$, the most interesting behavior for v as $(\varepsilon_1, \varepsilon_2) \to (0,0)$ occurs at $x = 0$, which we analyzed carefully in the last section. With our knowledge of $V(t) = v(0,t)$ from the last section, we can now express the vanishing viscosity limit, $v(x,t)$, where $x \neq 0$ as the solution to the initial/boundary value problem

$$\begin{aligned} v_t + (v_x)^2 - a^2 &= 0 \qquad x \neq 0,\ t > 0 \\ v(x,0) &= 0 \\ v(0,t) &= V(t), \end{aligned}$$

since it is clear that $v(x,t)$ has the following (deterministic) control theory representation:

$$v(x,t) = \min_{\alpha(\cdot,\cdot)\in M} \left[a^2\tau + \frac{1}{4}\int_0^\tau \alpha^2\left(y(s),s\right) ds + V(t-\tau) \right] \tag{40}$$

subject to the dynamics

$$\frac{dy(s)}{ds} = \alpha\left(y(s),s\right), \qquad y(t) = x,$$

where M is the set of all measurable functions and $\tau = \min[t, \inf\{s : y(s) = 0\}]$; that is, τ is the first time that the absolutely continuous path $y(s)$ intersects either the boundary or the "initial" condition (which, of course, is really the *final* condition from the control theory perspective).

When $\varepsilon_2 = o\left(\varepsilon_1^2\right)$ as $\varepsilon_1 \to 0$, $v(0,t) = V(t) = 0$, so the solution to (40) is

$$v(x,t) = \min\left[a^2 t, a\,|x|\right]. \tag{41}$$

This can be proven by first using Jensen's inequality to establish that the function $\alpha\left(y(s),s\right)$ which minimizes the right hand side of (40) must be equal to a constant. Simple calculus then establishes that $\alpha\left(y(s),s\right)=-2a$ if $x\in(0,at]$ in which case we are drawn to the boundary condition at $x=0$, $\alpha\left(y(s),s\right)=2a$ if $x\in[-at,0)$ in which case we are also drawn to the boundary condition at $x=0$, and $\alpha\left(y(s),s\right)=0$ if $|x|>at$, in which case we are too far from $x=0$ to be able to effectively take advantage of the cheap path on the boundary so instead the minimizing path is $y(s)=x$ which terminates on the "initial" condition. Note that there is a shock at $x=0$ and also at $x=\pm at$. The paths $y(s)$ that minimize (40) are characteristic curves, and we see that the characteristic curves enter the shocks at $x=\pm at$ (that is, they are Lax entropy shocks), but they actually exit the shock at $x=0$ (so this shock is an undercompressive wave for the v family of characteristics).

When $\varepsilon_1=O\left(e^{-\frac{a^2t}{\varepsilon_2}}\right)$ as $\varepsilon_2\to 0$, $v(0,t)=V(t)=a^2t$, so the solution to (40) is

$$v(x,t)=a^2t. \tag{42}$$

The stochastic noise has removed any advantage in approaching the $x=0$ boundary so instead the Lagrangian is minimized by selecting $\alpha\left(y(s),s\right)=0$ inducing the minimizing path $y(s)=x$, which terminates on the "initial" condition.

In conclusion, we have shown that the vanishing viscosity limit for our example is not unique as it depends upon how the viscosities vanish. If ε_1 stays much larger than ε_2 as both ε_1 and ε_2 vanish, then we obtain the solution (41). If ε_2 stays much larger than ε_1 as both ε_1 and ε_2 vanish, then we obtain the solution (42). If neither ε_1 nor ε_2 dominate as they vanish, then either 1) $V(t)$ is not defined and therefore the vanishing viscosity limit $v(x,t)$ is undefined or 2) $V(t)$ is defined and so we know from our rough bounds (27) and (30) that $0\le V(t)\le a^2t$, which implies, by (41), (42), and the maximum principle, that the vanishing viscosity limit will also be bounded; specifically,

$$\min\left[a^2t,a\left|x\right|\right]\le v(x,t)\le a^2t. \tag{43}$$

For example, in the case explored numerically in the previous section where $\varepsilon_1=\varepsilon_2\to 0$, we obtained evidence that $V(t)\approx 0.62a^2t$, so, by (40), the vanishing viscosity limit in this case is

$$v(x,t)=\min\left[a^2t,\sqrt{1-0.62}a\left|x\right|+0.62a^2t\right],$$

which, indeed, conforms to the bounds in (43).

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