Extending Viscosity Solutions to Eikonal Equations with Discontinuous Spatial Dependence

Daniel N. Ostrov

April 14, 1998

Abstract

The eikonal equation relates the known light intensity of a black and white photograph of a continuous surface to the surface's unknown three-dimensional shape. While it is known that a unique viscosity solution for the surface shape exists if the light intensity is a continuous function, little is known about reconstuctions for surfaces with kinks (i.e., gradient discontinuities) leading to discontinuous intensity fuctions.

We construct a unique stable solution for a very general set of discontinuous intensity functions by considering two families of continuous intensity functions which converge from above and below to the discontinuous function of interest. Using control theory we prove that the viscosity solutions corresponding to the two families must converge to the same function, and we define this unique limit to be the solution of interest. Finally, both continuity properties and control theory representations for this solution are established.

1 Introduction

A fundamental issue in the field of Optical Shape-From-Shading is how to determine a three dimensional shape using a two dimensional black and white photographic image of the shape. Consider a surface whose height is described by the continuous function $u(x, y)$. If the surface is illuminated from above (i.e., with light rays of uniform strength moving downward parallel to the z axis), the surface is Lambertian (i.e., the surface reflects incident light equally in all directions), and the albedo (that is, the grayness) of the surface is known (which allows the photographic image to be adjusted to correspond to a uniformly white surface), then the light intensity on the photograph, $I(x, y)$, is just the cosine of the angle between the positive z axis and the direction normal to the surface at (x, y). Expressing the cosine in terms of the partial derivatives of the height function yields

$$\frac{1}{\sqrt{1 + u_x^2 + u_y^2}} = I(x, y),$$

the Optical Shape-From-Shading equation. This equation can be re-expressed as the eikonal equation, which also arises in the study of geometric optics:

$$\sqrt{u_x^2 + u_y^2} - n(x, y) = 0 \tag{1}$$

where $n(x, y) = \sqrt{\frac{1}{I^2(x,y)} - 1}$ is a known function since $I(x, y)$ is known. Equation (1) will be the focus of this paper.

In the 1970's, Horn [1] and many others analyzed (1) using the fact that the equation's characteristic curves propagate in the direction of the surface gradient, Du. By 1992, Dupuis and Oliensis [2] had used control theory to establish that if u is assumed to be smooth and $n = 0$ at only a finite, but non-zero, number of (x, y) locations, then knowledge of the value and concavity of u at one of these locations generally allows u to be determined. This exploited the fact that when u is smooth, the characteristics cannot collide, so they must always travel to and from the locations where $n = 0$ (i.e., the *critical points* of the associated phase plane flow).

If n is continuous but it cannot be assumed that u is smooth, then we need more than knowledge at a single critical point to determine a unique solution. By applying a result of Ishii's [3] concerning the notion of the *viscosity solution* developed by Crandall, Lions [4] and others in the 1980's, Rouy and Tourin [5] established that if u is specified ($u = g$) on $\partial\Omega$ where Ω is an open, bounded subset of the (x, y) plane where (1) is defined and n is positive, then (1) has, at most, one viscosity solution. (In practice this generally means u is specified both on the edge of some simply connected region and at all of the critical points within the region.) Also, in [6], Lions, Rouy and Tourin explored viscosity solutions for Optical Shape-From-Shading problems where shadows can occur on the surface.

The standard definition of a viscosity solution can be found in many sources (originally [4] and [7]). For (1) where n is continuous, the definition corresponds to the following straightforward geometric interpretation: Assuming — as we will throughout this paper — that n has an upper bound, B, the viscosity solution for (1) must be a Lipschitz solution, as we establish in section 4, and this Lipschitz solution is characterized by possibly having kinks that point upward, but never kinks pointing downward. Specifically, at any location (x_0, y_0) where the viscosity solution, $u(x, y)$, is not differentiable, one can find smooth surfaces that touch the point $(x_0, y_0, u(x_0, y_0))$ and (locally) lie above the surface $u(x, y)$, but no smooth surface can exist that touches $(x_0, y_0, u(x_0, y_0))$ and lies below the surface $u(x, y)$. Since the left hand side of (1) is convex in Du, the viscosity solution also has a control theory representation [5], [8] which will be more useful for the purposes of this paper. (The control theory representation will be reviewed in section 2.3).

There are two reasons to select the viscosity solution as the unique solution of interest. The first reason is that the viscosity solution is continuous on $\partial\Omega$ if continuity is possible. In other words, if the viscosity solution to (1) is not continuous on $\partial\Omega$, then no weak Lipschitz solution to (1) exists that is continuous on $\partial\Omega$. The second reason is that the viscosity solution is often the smoothest possible solution, in which case it is the most likely solution to correspond to the actual surface. For example, consider the case where Ω is a square, $u = 0$ on $\partial\Omega$, and $n = 1$ in Ω. The surface corresponding to the viscosity solution for this case is a square pyramid pointing upwards. Alternative Lipschitz solutions can be created by taking the section of the pyramid where u is greater than or equal to any positive constant and inverting it, but these solutions are less smooth and less likely to correspond to the actual surface. (We point out, however, that sometimes there are clear examples of equally smooth nonviscosity solutions—such as the completely inverted square pyramid, which points downwards in the above example—and there is also evidence suggesting that, for some examples, nonviscosity solutions may exist that are more smooth than the viscosity solution [9].)

The study of nonsmooth surfaces corresponding to continuous n functions is of limited practical use in Shape-From-Shading since nonsmooth surfaces almost always correspond to discontinuous n functions. In [10], Ishii defined an extension of viscosity solutions to some Hamiltonians with discontinuous dependence on spatial variables. Using Ishii's definition, Tourin, in [11], was able to show that (1)has a unique Ishii solution even if n is discontinuous when n conforms to the following restrictions which are necessary, in part, because the Ishii solution can only contain kinks that point upward: 1) n must be Lipschitz continuous except along a single smooth curve of (possible) discontinuity that divides Ω into two regions, R_1 and R_2, 2) There can be no critical points in Ω and n must be bounded away from 0 near the curve of discontinuity, 3) At any point on the discontinuity curve, the limit of the intensity if the point is approached from within the region R_1 is greater than or equal to the limit of the intensity if the point is approached from within the region R_2, and 4) $g = 0$ along the entire boundary of Ω.

In this paper we introduce a new notion for uniquely extending the viscosity solution to (1) with discontinuous n which is applicable to almost any n with a finite number of curves and points of discontinuity. This notion of solution allows for kinks that point downward at locations where n is discontinuous— including allowing characteristics to propagate *out* from shocks. Also, as shown in [12], this notion of solution can be computed by explicit convergent numerical schemes.

The new notion of solution is physically motivated. Because the notion of a physical surface involves macroscopic averaging of the microscopic behavior of the surface and because photographic data contains some bleedthrough from the neighborhood of any location, we want to consider continuous n functions derived from locally perturbing the discontinuous n of interest. We will define n^ε and n_ε as continuous functions that converge from above and below, respectively, to the discontinuous n of interest; the n^ε and n_ε will agree with the discontinuous n except within the ε neighborhood of n's discontinuities. Because the n^ε and n_ε are continuous, they have corresponding viscosity solutions, which we will call u^ε and u_ε. Our goal is to try to define u, the solution corresponding to the discontinuous n, as the common limit of the u^ε and u_ε as $\varepsilon \to 0$.

To establish the main theorem of this paper, which states that this common limit exists for a very general class of discontinuous n, we start in the next section with the background for our theorem, which will entail stating our assumptions, explicitly defining the n^ε and n_ε, reviewing the control theory representation of the viscosity solution, and explicitly stating the main theorem. In section 3, we prove the main theorem. In section 4, we show that when u, the common limit, is defined, it must be a Lipschitz continuous function and, if the u^ε are continuous on $\partial\Omega$, u is continuous on $\partial\Omega$. Finally, we establish a control theory representation for u.

2 Formation and Background of the Main Theorem

2.1 Statement of Assumptions

We make the following assumptions concerning 1) the domain Ω, 2) the function $n(X)$ where $X \equiv (x, y)$, and 3) the region $J \subset \overline{\Omega}$, which is the closure of the region where n, if it were defined throughout Ω, would be discontinuous:

1) Ω is an open, bounded region in $\mathbf{R}^2$.

2) $n(X)$ is defined for all $X \in \Omega - J$ and

a) There exists a finite B such that $0 < n(X) \leq B$ for all $X \in \Omega - J$.

b) n is locally Lipschitz with Lipschitz constant L on $\Omega - J$. Specifically,

$$|n(X) - n(Y)| \leq L\hat{d}(X, Y),$$

where $\hat{d}(X, Y) \equiv \inf_{C \in \hat{\aleph}} \text{arclength}(C)$ and $\hat{\aleph}$ is the set of all arcs $\subset \Omega - J$ that connect X and Y.

3) $J \subset \overline{\Omega}$ is comprised of the union of a finite number (N) of non-self-intersecting (i.e., simple and non-closed) C^2 curves and points labelled $\Gamma_1, \Gamma_2, ..., \Gamma_N$ that conform to the following:

a) We define $\gamma_i(t)$ as the parameterization by arclength t of locations on Γ_i :

$$\Gamma_i \equiv \{\gamma_i(t) : t \in [0, T_i]\},$$

and assume that $T_i < \infty$, so each Γ_i has finite (possibly zero) arclength T_i. (We note that if any Γ_i is a point (i.e., $T_i = 0$), then assumption (2b) must be relaxed as the point is approached. We accommodate this—and more—by allowing L to become infinite as any endpoint (i.e., $\gamma_i(0)$ or $\gamma_i(T_i)$) is approached. This means that L is a function of δ, where δ is defined in subsection 3.2; specifically L, the Lipschitz constant on $\Omega - J -$ the union of δ neighborhoods around each endpoint, can only become infinite as δ shrinks to zero.) We also assume that $\{\gamma_i(t) : t \in (0, T_i)\} \subset \Omega$; that is, only the endpoints can be on $\partial\Omega$.

b) We assume that Γ_i and Γ_j $(i \neq j)$ can only intersect at end points, i.e.,

$$\gamma_i(t_i) = \gamma_j(t_j) \Rightarrow t_i \in \{0, T_i\} \text{ and } t_j \in \{0, T_j\}.$$

c) Further, if Γ_i and Γ_j meet, we assume that they meet at a nonzero angle.

d) The limit of n as we approach J is assumed to be nonzero, i.e., $\forall Y \in J$, $\liminf_{X \to Y} n(X) \neq 0$.

e) We also assume that the limit of n from one side of Γ_i (we will use Γ_i^+ to refer to this side of the curve) is greater at any point on Γ_i than the limit of n from the other side of Γ_i (we will refer to this side of the curve as Γ_i^-); that is,

$$\lim_{X \in \Gamma_i^+ \to \gamma_i(t)} n(X) > \lim_{X \in \Gamma_i^- \to \gamma_i(t)} n(X) \quad \forall t \in (0, T_i) \quad i = 1, 2, ..., N.$$

Note that at the endpoints, we specify no additional restriction here; $\lim_{X \to \gamma(0)} n(X)$ and $\lim_{X \to \gamma(T_i)} n(X)$ may or may not exist.

f) Finally, near endpoints of the Γ_i on $\partial\Omega$, we assume $\partial\Omega$ is not pathological; specifically, defining $d(X, Y)$ to be the standard l_2 distance between points X and Y and defining the "ball" $B_\Delta(X)$ to be the set of all points $Y \in \Omega$ where $d(X, Y) < \Delta$, we assume that there exists constants ρ and κ such that for any endpoint p on $\partial\Omega$, any positive $r \leq \rho$, and any points $X, Y \in B_r(p)$, there exists an arc contained *within* Ω of arclength less than or equal to κr that connects X and Y. (We further make $\kappa \geq 2$, which guarantees that this property also applies to endpoints p not on $\partial\Omega$.)

Assumptions (3a)–(3f) are not very restrictive; it would be quite unusual for a surface to have discontinuities where Γ_i could not be defined within these requirements. To demonstrate the general nature of J satisfying our assumptions, we refer to Figure 1. Note that a curve of discontinuity may fail to be C^2 as long as this failure is only at a finite number of locations allowing us to re-express the curve as a union of C^2 curves (e.g., Γ_4 and Γ_5 in the figure). Similarly, we can have curves of discontinuity intersect as long as the number of intersections is finite (e.g., Γ_5, Γ_6, and Γ_{12} or Γ_6, Γ_7, Γ_8, and Γ_9). Also, the side of the curve of discontinuity where n is greater may switch as

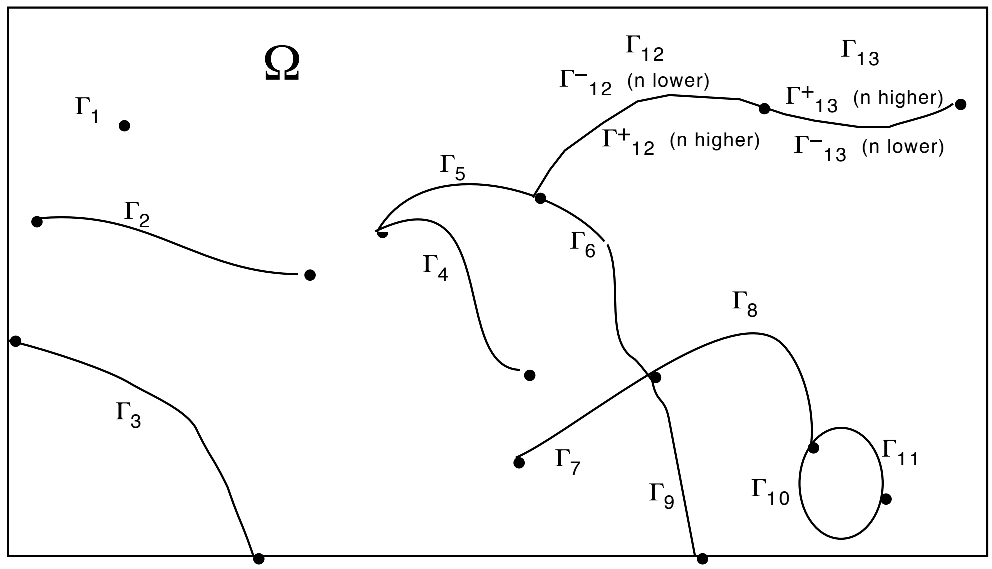

Figure 1: *The nature of the discontinuities* Γ_i *in the domain* Ω *is quite general. The* Γ_i *may be points or curves that may or may not intersect each other or* $\partial\Omega$ *at endpoints.*

we progress down the curve; we merely require the number of these switches to be finite (e.g., Γ_{12} and Γ_{13}). Finally, we note that surfaces violating assumption (3d) almost never occur; if they do, almost any arbitrarily small rotation of the surface will create a surface where assumption (3d) is justified again.

2.2 Definition of n^ε and n_ε, the Approximate n Functions

Define $Z(X)$ for $X \in \Omega$ to be a point on J that minimizes the distance between X and J (i.e., $Z(X) \in J$ such that $d(X, Z(X)) = \min_{Y \in J} d(X, Y)$).

Now, for every $X \in \Omega - J$, define

$$\tilde{n}^{\varepsilon}(X) \equiv \begin{cases} \limsup_{Y \to Z(X)} n(Y) & \text{if } d(X, Z(X)) \leq \varepsilon \\ & \text{and } \lim_{\alpha \to 0^+} n(Z(X) + \alpha(X - Z(X))) < \limsup_{Y \to Z(X)} n(Y) \\ n(X) & \text{otherwise.} \end{cases}$$

(Basically, the value of n on $\cup_{i=1}^{N} \Gamma_i^-$ within ε of J has been raised to form $\tilde{n}^{\varepsilon}$.) Now we define $n^{\varepsilon}(X)$ to be any continuous function on Ω such that for each $X \in \Omega - J$, $n(X) \leq n^{\varepsilon}(X) \leq \tilde{n}^{\varepsilon}(X)$ (see Figure 2) and if $\varepsilon_1 > \varepsilon_2$ then $n^{\varepsilon_1}(X) \geq n^{\varepsilon_2}(X)$. Similarly, we define

$$\tilde{n}_{\varepsilon}(X) \equiv \begin{cases} \liminf_{Y \to Z(X)} n(Y) & \text{if } d(X, Z(X)) \leq \varepsilon \\ & \text{and } \lim_{\alpha \to 0^+} n(Z(X) + \alpha(X - Z(X))) > \limsup_{Y \to Z(X)} n(Y) \\ n(X) & \text{otherwise} \end{cases}$$

and $n_{\varepsilon}(X)$ to be any continuous function on Ω such that for each $X \in \Omega - J$, $n(X) \geq n_{\varepsilon}(X) \geq \tilde{n}_{\varepsilon}(X)$ (again, see Figure 2) and if $\varepsilon_1 > \varepsilon_2$ then $n_{\varepsilon_1}(X) \leq n_{\varepsilon_2}(X)$. For example, if $\varepsilon < \frac{1}{L}$ (L is the Lipschitz constant of n from assumption (2b)), we could define $n^{\varepsilon}(X) = \sup_{Y \in \Omega - J}(n(Y) - \frac{B}{\varepsilon} d(Y, X))$ and $n_{\varepsilon}(X) = \inf_{Y \in \Omega - J}(n(Y) + \frac{B}{\varepsilon} d(Y, X))$.

Note that $n^{\varepsilon} \searrow n$ and $n_{\varepsilon} \nearrow n$ for all $X \in \Omega - J$ as $\varepsilon \to 0$.

2.3 Control Theory formulation of the Viscosity Solution

We now review the control theory form of a viscosity solution [5], [8]. If $u(X)$ is the (unique) bounded, uniformly continuous viscosity solution of any Hamilton-Jacobi equation with Dirichlet boundary conditions:

$$\begin{aligned} H(X, Du(X)) &= 0, \ X \in \Omega \\ u(X) &= g(X), \ X \in \partial\Omega \end{aligned}$$

where the *Hamiltonian*, H, is a continuous function that is convex in Du, then $u(X)$ can be represented by

$$u(X) = \inf_{\alpha(\cdot) \in A} \left[\int_0^S h(x(s), \alpha(s)) ds + g(x(S)) \right]$$

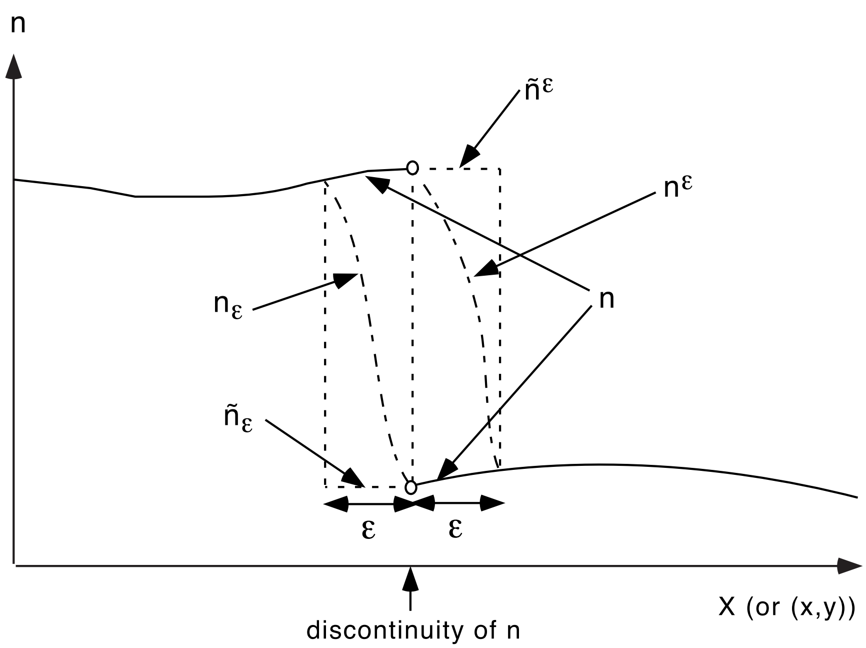

Figure 2: *The discontinuous intensity n (represented by the solid line) is approximated by the continuous functions n^ε and n_ε (represented by alternating long and short dashes). The approximations conform to the bounds $n \le n^\varepsilon \le \tilde{n}^\varepsilon$ and $n \ge n_\varepsilon \ge \tilde{n}_\varepsilon$.*

subject to the dynamics

$$\dot{x}(s) = f(x(s), \alpha(s)); \quad x(0) = X, \; x(S) \in \partial\Omega,$$

where A is the set of all measurable functions from $[0, S]$ to C, a compact set in $\mathbf{R}^2$, and $h : \mathbf{R}^2 \times C \to \mathbf{R}$ and $f : \mathbf{R}^2 \times C \to \mathbf{R}^2$ are any functions satisfying

$$H(x, p) = -\min_{a \in C}\{f(x, a) \cdot p + h(x, a)\}. \tag{2}$$

Now we let $C = S^1$, the set of all unit vectors in $\mathbf{R}^2$, and choose $f(x, a) = a$ and $h(x, a) = \hat{n}(x)$ where $\hat{n}(x)$ is continuous. The minimum of (2) is attained at $a = \frac{-p}{\|p\|}$, which leads to $H(x, p) = \|p\| - \hat{n}(x)$, the Hamiltonian for the eikonal equation. Therefore, any bounded, uniformly continuous viscosity solution to (1) — if $\hat{n}(x)$ is continuous — can be represented by

$$u(X) = \inf_{\alpha(\cdot) \in A}\left[\int_0^S \hat{n}(x(s))ds + g(x(S))\right] \tag{3}$$

subject to the dynamics

$$\dot{x}(s) = \alpha(s) \text{ where } x(0) = X, \; x(S) \in \partial\Omega, \text{ and } \|\alpha(s)\| = 1 \; \forall s \in [0, S]. \tag{4}$$

We note that the notion of viscosity solution presented here is not necessarily continuous at the boundary of Ω.

2.4 Statement of the Main Theorem

Let $u^\varepsilon(X)$ be the viscosity solution defined by (3) and (4), where $\hat{n}$ is the continuous function n^ε, and let $u_\varepsilon(X)$ be the viscosity solution defined by (3) and (4), where $\hat{n}$ is the continuous function n_ε. Our goal in the next section is to prove that

$$\lim_{\varepsilon \to 0} u^\varepsilon(X) = \lim_{\varepsilon \to 0} u_\varepsilon(X) \qquad \forall X \in \Omega.$$

Once this is proven, we can *define* the unique solution to (1) with discontinuous $n(x)$ as this common limit.

3 Proof of the Main Theorem

Subsection 3.1 is devoted to showing that $u^\varepsilon \geq u_\varepsilon$ and that $\lim_{\varepsilon\to 0} u^\varepsilon$ and $\lim_{\varepsilon\to 0} u_\varepsilon$ exist, thereby establishing that $\lim_{\varepsilon\to 0} u^\varepsilon \geq \lim_{\varepsilon\to 0} u_\varepsilon$. Subsections 3.2, 3.3, and 3.4 establish the much more difficult inequality: $\lim_{\varepsilon\to 0} u^\varepsilon \leq \lim_{\varepsilon\to 0} u_\varepsilon$.

3.1 Monotonicity of solutions

Theorem 1 *Let $u_i(X)$ $(i = 1, 2)$ be defined by* (3) *and* (4)*, where $\hat{n}(x(s))$ is replaced by arbitrary continuous functions $n_i(x(s))$. If $n_1(X) \geq n_2(X)$ $\forall X \in \Omega$, then $u_1(X) \geq u_2(X)$ for any $X \in \Omega$.*

Proof. Consider any $X \in \Omega$; we know from (3) and (4) that

$$u_1(X) = \inf_{\alpha(\cdot)\in A} \left[\int_0^S n_1(x(s))ds + g(x(S)) \right]$$

where $\dot{x}\,(s) = \alpha(s)$ and $A = \{\alpha : [0, S] \to S^1 \mid \alpha(\cdot) \text{ is measurable}\}$. Therefore, there exists a sequence $\alpha_i(\cdot) \in A$ $(i = 1, 2, ...)$ such that

$$u_1(X) = \lim_{i\to\infty} \left[\int_0^{S_i} n_1(x_i(s))ds + g(x_i(S_i)) \right]$$

where $\dot{x}_i\,(s) = \alpha_i(s)$. Since $n_1(X) \geq n_2(X)$, we have that

$$\begin{aligned} \lim_{i\to\infty} \left[\int_0^{S_i} n_1(x_i(s))ds + g(x_i(S_i)) \right] &\geq \lim_{i\to\infty} \left[\int_0^{S_i} n_2(x_i(s))ds + g(x_i(S_i)) \right] \\ &\geq \inf_{\alpha(\cdot)\in A} \left[\int_0^S n_2(x(s))ds + g(x(S)) \right] \\ &= u_2(X). \end{aligned}$$

Therefore, $u_1(X) \geq u_2(X)$. ■

Corollary 2 $u^\varepsilon(X) \geq u_\varepsilon(X)$.

Proof. The corollary follows directly from Theorem 1 since $n^\varepsilon(X) \geq n_\varepsilon(X)$. ■

Corollary 3 $\lim_{\varepsilon\to 0} u^\varepsilon(X)$ *and* $\lim_{\varepsilon\to 0} u_\varepsilon(X)$ *exist.*

Proof. Since $n^\varepsilon(X)$ and $n_\varepsilon(X)$ are each monotone in ε, $u^\varepsilon(X)$ and $u_\varepsilon(X)$ must also each be monotone in ε by Theorem 1. Since both families are bounded, the limits must exist. ■

3.2 Restrictions on the δ regions surrounding the endpoints of the curves of discontinuity

We will show that $\|u^\varepsilon - u_\varepsilon\|_{L^\infty(\Omega)}$ is small by showing that, at any point X, the associated infimum of costs for u^ε and u_ε are similar. Since $n^\varepsilon(X) = n_\varepsilon(X)$ except possibly for X within an ε neighborhood of J, the difference between u^ε and u_ε is dependent upon how often the (approximately) cost minimizing paths of u^ε and u_ε stay within the ε neighborhood of J. We will show that what little advantage in cost savings can be obtained by staying near J must vanish as $\varepsilon \to 0$. The analysis of the cost accrued near endpoints of each Γ_i will differ from the analysis of the cost accrued near the interior of each Γ_i. We will cover the endpoints with disks of arbitrarily small radius, $\delta > 0$, and use the upper bound, B, on the cost function and the small size of these δ disks to bound the cost difference accrued within these disks. Also, for sufficiently small ε, we will analyze the cost difference accrued near the interiors of the Γ_i and show that $\|u^\varepsilon - u_\varepsilon\|_{L^\infty(\Omega)} \le f(\delta, \varepsilon)$ where $\lim_{\delta\to 0} \lim_{\varepsilon\to 0} f(\delta, \varepsilon) = 0$.

We begin by covering the endpoints. Define P to be the set containing the endpoints of all the Γ_i; i.e.,

$$P \equiv \cup_{i=1}^{N} \{\gamma_i(0), \gamma_i(T_i)\}.$$

We will use p to denote an arbitrary endpoint. For convenience we define $B_\delta(P) \equiv \cup_{p\in P} B_\delta(p)$, the collection of δ neighborhoods of the endpoints.

We only consider $\delta \in (0, \delta_0]$ where the bound, δ_0, is a positive number that can be reduced as needed since we are only interested in arbitrarily small δ. Further, we will only consider ε such that $0 < \varepsilon \le \varepsilon_0(\delta)$, where ε_0 is an arbitrarily small positive monotone increasing function that we can restrict (specifically, reduce) as needed since we are only interested in the behavior of u^ε and u_ε as $\varepsilon \to 0$.

We now begin to restrict δ_0 and ε_0.

First, assumption (3b) guarantees that we can restrict δ_0 small enough so that $\forall\, 0 < \delta \le \delta_0$,

$$B_{3\delta}(p) \cap \Gamma_i \ne \emptyset \;\;\Rightarrow\;\; \begin{array}{l}\text{(a) } p = \gamma_i(0) \text{ or } \gamma_i(T_i) \text{ and} \\ \text{(b) } \partial B_{3\delta}(p) \cap \Gamma_i \text{ consists of exactly one point,}\end{array}$$

which implies that (1) the δ disks cannot overlap, (2) the 3δ disks (and therefore also the δ disks) only contain Γ_i whose endpoints are at the center of the disk, and (3) these Γ_i do not weave in and out of the 3δ (or δ) disk.

Next we restrict δ_0 and ε_0 so that $B_{\delta_0}(P)$ and the ε neighborhood of J stay away from areas of arbitrarily small cost. Both J and $Z \equiv \{X \in \overline{\Omega} : \liminf_{Y \to X} n(Y) = 0\}$, the set of zero cost, are closed sets. Since, by assumption (3d), $\liminf_{Y \to X} n(Y) \neq 0 \, \forall Y \in J$, we have that $Z \cap J = \emptyset$. Therefore, there exists a closed region Ω' and scalars n_0 and Δ such that

$$1) \; J \subset \Omega' \subset \overline{\Omega}$$

$$2) \inf_{X \in \Omega' - J} n(X) \geq n_0 > 0$$

$$3) \; d(J, \partial\Omega' - \partial\Omega) \geq \Delta > 0.$$

Now we restrict $\varepsilon_0 < \delta_0 < \Delta$, so $B_{\delta_0}(P)$ and the ε neighborhood of J stay at least Δ distance away from any point where $n < n_0$, which will be of use in the next subsection.

By assumptions (3b) and (3c), Γ_i and Γ_j can only intersect at their endpoints, and if they do intersect, the angle of intersection must be nonzero. Therefore, we can define $\theta \in (0, 1]$ to be an angle that is strictly smaller than any angle of intersection between any Γ_i and Γ_j $(i, j = 1, ..., N)$. Since, by assumption, all $\Gamma_i \in C^2$, any Γ_i and Γ_j that intersect at p, must — within $B_\delta(p)$ — become progressively like two line segments intersecting at an angle $> \theta$ as δ decreases. Therefore, if δ_0 is small enough, for any $0 < \delta \leq \delta_0$,

$$\min_{\substack{i,j,p \\ i \neq j}} d(\partial B_\delta(p) \cap \Gamma_i, \partial B_\delta(p) \cap \Gamma_j) \geq \delta \sin\left(\frac{\theta}{2}\right), \tag{5}$$

so, for sufficiently small δ_0, we have from (5) and assumption (3b) that for any $0 < \delta \leq \delta_0$,

$$\begin{aligned} D_\delta \;\; &\equiv \;\; \min_{\substack{i,j \\ i \neq j}} d(\Gamma_i - B_\delta(P), \Gamma_j - B_\delta(P)) \geq \delta \sin\left(\frac{\theta}{2}\right) \text{ and} \\ & \quad\;\; \min_{\substack{i,j \\ i \neq j}} d(\Gamma_i - B_\delta(P), \Gamma_j) \geq \frac{\delta}{2} \sin(\theta) . \end{aligned} \tag{6}$$

Finally, we restrict $\delta_0 \leq \rho$ where ρ is defined in assumption (3f). These restrictions will be exploited in the next subsection.

3.3 Analysis of ε neighborhoods near the interiors of the curves of discontinuity

Throughout this subsection we will think of δ as fixed and we will concentrate on shrinking $\varepsilon_{0.}$ We begin with a number of definitions that enable us to discuss the interiors of the Γ_i, small continuous sections within a partition of the interiors of the Γ_i, and ε neighborhoods of the these continuous sections.

We are primarily interested in the behavior near $\Gamma_{i,\delta}$, which is the part of Γ_i not near its endpoints (i.e., the interior of Γ_i) which we define by

$$\Gamma_{i,\delta} \equiv \Gamma_i - B_{\frac{3}{4}\delta}(P).$$

To analyze this region we will need to also consider a slightly larger interior region of Γ_i, which we will distinguish using an asterisk:

$$\Gamma_{i,\delta^*} \equiv \Gamma_i - B_{\frac{1}{2}\delta}(P).$$

Analogous to $J \equiv \cup_{i=1}^{N} \Gamma_i$, we define

$$\begin{aligned} J_\delta &\equiv \cup_{i=1}^{N} \Gamma_{i,\delta} \\ J_{\delta^*} &\equiv \cup_{i=1}^{N} \Gamma_{i,\delta^*}. \end{aligned}$$

In the analysis of J_{δ^*} we will later make use of the definition

$$\Delta n_{\min} \equiv \min_{X \in J_{\delta^*}} \left(\limsup_{Y \to X} n(Y) - \liminf_{Y \to X} n(Y) \right) > 0.$$

(That $\Delta n_{\min} > 0$ follows from J_{δ^*} being a closed, bounded set and assumption (3e).)

We are particularly interested in small continuous sections of $\Gamma_{i,\delta}$ with arclength $\lambda > 0$; we will analyze these sections by containing them within small continuous sections of Γ_{i,δ^*} of arclength λ^* which we will shrink as needed. We will use $\gamma(t)$ to describe the parameterization of subsections of Γ_i of interest. If $\gamma(t)$ is used in the context of a λ section of $\Gamma_{i,\delta}$, then we are considering an arbitrary i and t_0 where $\gamma(t) = \gamma_i(t_0 + t) \in \Gamma_{i,\delta}$ for $t \in [0, \lambda]$; if $\gamma(t)$ is used in the context of a λ^* section of Γ_{i,δ^*}, then we are considering an arbitrary i and t_0 where $\gamma(t) = \gamma_i(t_0 + t) \in \Gamma_{i,\delta^*}$ for $t \in [0, \lambda^*]$.

Next we define $\overrightarrow{n}(\gamma_i(t))$ to be the unit vector normal to Γ_i at $\gamma_i(t)$ pointing in the direction Γ_i^- (i.e., towards the lower n values); this will allow us to discuss regions near $\Gamma_{i,\delta}$ and Γ_{i,δ^*}. Let $\gamma(t)$ be a λ section of $\Gamma_{i,\delta}$ so we can define

$$\Gamma \equiv \cup_{t\in[0,\lambda]}\gamma(t),$$

the region

$$\Gamma_\varepsilon \equiv \cup_{\alpha\in[-\varepsilon,\varepsilon]} \cup_{t\in[0,\lambda]} \left[\gamma(t) + \alpha \overrightarrow{n}(\gamma(t))\right],$$

and the four sides of the Γ_ε region:

$$\Gamma_\varepsilon^- \equiv \cup_{t\in[0,\lambda]} \left[\gamma(t) + \varepsilon \overrightarrow{n}(\gamma(t))\right],$$

$$\Gamma_\varepsilon^+ \equiv \cup_{t\in[0,\lambda]} \left[\gamma(t) - \varepsilon \overrightarrow{n}(\gamma(t))\right],$$

Γ_ε^0, and $\Gamma_\varepsilon^\lambda$ where

$$\Gamma_\varepsilon^t \equiv \cup_{\alpha\in[-\varepsilon,\varepsilon]} \left[\gamma(t) + \alpha \overrightarrow{n}(\gamma(t))\right].$$

Similarly, for $\gamma(t)$ describing a λ^* section of Γ_{i,δ^*}, define Γ_{λ^*},the region $\Gamma_{\lambda^*,\varepsilon}$, and its four sides $\Gamma_{\lambda^*,\varepsilon}^-$, $\Gamma_{\lambda^*,\varepsilon}^+$, $\Gamma_{\lambda^*,\varepsilon}^0$, and $\Gamma_{\lambda^*,\varepsilon}^{\lambda^*}$ as above except with λ replaced by λ^*.

Now we restrict ε_0 so that the 2ε neighborhoods of the Γ_{i,δ^*} regions do not intersect. Specifically, we first restrict $\varepsilon_0 < \frac{\delta\theta}{16}$ so that $D_{\frac{\delta}{2}} > 4\varepsilon_0$ and therefore the 2ε neighborhoods of Γ_{i,δ^*} and Γ_{j,δ^*} $(i \neq j)$ cannot overlap. Further, this restriction implies that all points X where $n^\varepsilon(X) \neq n_\varepsilon(X)$ are either elements of $B_\delta(P)$ and/or one of the Γ_ε regions.

Next we want to restrict ε_0 so that, for any i, the 2ε neighborhood of Γ_i cannot overlap itself. This requires controlling the effect of the curvature of Γ_i, and so we define $R > 0$ to be the minimum radius of curvature for all of the Γ_i. (R is defined and nonzero because, by assumption, each Γ_i is C^2, closed, and of finite arclength.) Now restrict ε_0 so that

$$4\varepsilon_0 < \min\{d(\gamma_i(t_1), \gamma_i(t_2)) : i = 1, ..., N;\ t_1, t_2 \in [0, T_i];\ \text{and}\ |t_1 - t_2| \geq \pi R\}. \tag{7}$$

(By assumption, no Γ_i can intersect itself, therefore the right hand side of (7) must be nonzero.)

Finally, restrict λ^* and ε_0 so that the convex hull of any $\Gamma_{\lambda^*,\varepsilon}$ (or Γ_ε) is a subset of Ω, which is possible since $J_{\delta^*} \subset \Omega$ by assumption (3a).

We are now ready to begin our analysis. Consider $x_\varepsilon(s)$, an (almost) minimizing path for $u_\varepsilon(X)$; specifically, let $x_\varepsilon(s)$ be a path such that

$$u_\varepsilon(X) - \left[\int_0^S n_\varepsilon(x_\varepsilon(s))ds + g(x_\varepsilon(S))\right] \geq -\epsilon(\varepsilon) \tag{8}$$

where $\epsilon(\varepsilon)$ is small and positive. We wish to analyze the possible behavior of $x_\varepsilon(s)$ should $x_\varepsilon(s)$ try to weave in and out of any $\Gamma_{\lambda^*,\varepsilon}$. The following lemma shows that for sufficiently small ε_0 and λ^*, if $x_\varepsilon(s)$ is at a point in $\Gamma^+_{\lambda^*,\varepsilon}$, it cannot stray more than a multiple of ε distance from $\Gamma_{\lambda^*,\varepsilon}$ if it hopes to return back to $\Gamma^+_{\lambda^*,\varepsilon}$; specifically, it cannot take values $\gamma(t) - \alpha\overrightarrow{n}(\gamma(t))$ where $\alpha \geq k\varepsilon$. This is accomplished by choosing λ^* small enough to reduce the effect of curvature and then comparing the cost of candidate $x_\varepsilon(s)$ paths with other paths (denoted by $y(s)$) that stay on $\Gamma^-_{\lambda^*,\varepsilon}$, where the cost is low.

Lemma 4 *For sufficiently small λ^* and ε_0, if (1) $x_\varepsilon(s_1)$ and $x_\varepsilon(s_2) \in \Gamma^+_{\lambda^*,\varepsilon}$ where $s_1 < s_2$ and (2) for each $s \in [s_1, s_2]$, there exists a $t \in [0, \lambda^*]$ and $\alpha \geq \varepsilon$ where*

$$x_\varepsilon(s) = \gamma(t) - \alpha\overrightarrow{n}(\gamma(t)),$$

then, defining $k \equiv \frac{20B^2}{n_0 \Delta n_{\min}}$, for no $s \in [s_1, s_2]$ can there exist a $t \in [0, \lambda^*]$ and $\alpha \geq k\varepsilon$ where

$$x_\varepsilon(s) = \gamma(t) - \alpha\overrightarrow{n}(\gamma(t)).$$

Proof. Assume for the moment that $x_\varepsilon(s)$ does not, for any $s \in [s_1, s_2]$, intersect the ε neighborhood of J.

Define $n_i = n(x_\varepsilon(s_i))$ where $i = 1, 2$; define t_i to be the value of $t \in [0, \lambda^*]$ such that $x_\varepsilon(s_i) = \gamma(t) - \varepsilon\overrightarrow{n}(\gamma(t))$ where $i = 1, 2$.

Since $\left\| \frac{dx_\varepsilon}{ds} \right\| = 1$, we can exploit the Lipschitz constant L from assumption (2b) to see that

$$n_\varepsilon(x_\varepsilon(s)) \geq n_{\varepsilon,small}(x_\varepsilon(s)) \equiv \begin{cases} 0 \vee [n_1 - L(s - s_1)] & \text{if } s_1 \leq s \leq (s^* \wedge s_2) \\ 0 \vee [n_2 + L(s - s_2)] & \text{if } (s_1 \vee s^*) \leq s \leq s_2 \end{cases}$$

where $s^* \equiv \frac{1}{2}\left(\frac{n_1 - n_2}{L} + s_1 + s_2\right)$. Now consider the path, $x_{\varepsilon,line}(s)$, going directly in a straight line from $x_\varepsilon(s_1)$ to $x_\varepsilon(s_2)$; i.e.,

$$x_{\varepsilon,line}(s) \equiv x_\varepsilon(s_1) + \frac{x_\varepsilon(s_2) - x_\varepsilon(s_1)}{\|x_\varepsilon(s_2) - x_\varepsilon(s_1)\|}(s - s_1) \tag{9}$$

where $s \in [s_1, s_2']$ and $s_2' \equiv s_1 + \|x_\varepsilon(s_2) - x_\varepsilon(s_1)\|$. Further, define

$$n_{\varepsilon,smaller}(x_{\varepsilon,line}(s)) \equiv \begin{cases} n_1 - L(s - s_1) & \text{if } s_1 \leq s \leq (s'^* \wedge s_2') \\ n_2 + L(s - s_2) & \text{if } (s_1 \vee s'^*) \leq s \leq s_2' \end{cases} \tag{10}$$

where $s'^{*} \equiv \frac{1}{2}\left(\frac{n_1-n_2}{L}+s_1+s'_2\right)$. Since $s'_2 \leq s_2$ we can bound the costs

$$\int_{s_1}^{s_2} n_\varepsilon(x_\varepsilon(s))ds \geq \int_{s_1}^{s_2} n_{\varepsilon,small}(x_\varepsilon(s))ds \geq \int_{s_1}^{s'_2} n_{\varepsilon,smaller}(x_{\varepsilon,line}(s))ds. \quad (11)$$

Substituting (10) into the right hand side of (11) and integrating, we have that

$$\int_{s_1}^{s_2} n_\varepsilon(x_\varepsilon(s))ds \geq \begin{cases} n_2(s'_2-s_1)-\frac{L}{2}(s'_2-s_1)^2 \text{ if } s'^{*} \leq s_1 \\ n_1(s'^{*}-s_1)+n_2(s'_2-s'^{*}) \\ \qquad -\frac{L}{2}\left((s'^{*}-s_1)^2+(s'_2-s'^{*})^2\right) \text{ if } s_1 \leq s'^{*} \leq s'_2 \\ n_1(s'_2-s_1)-\frac{L}{2}(s'_2-s_1)^2 \text{ if } s'^{*} \geq s'_2. \end{cases} \quad (12)$$

We note that (1) if $s'^{*} \leq s_1$, then $n_2 \geq n_1$, and therefore $n_2 \geq \frac{n_1+n_2}{2}$, that (2) if $s'^{*} \geq s'_2$, then $n_1 \geq n_2$, and therefore $n_1 \geq \frac{n_1+n_2}{2}$, and that (3) substituting the definition of s'^{*} into the right hand side of (12) for the case $s_1 \leq s'^{*} \leq s'_2$, we see that it equals $\frac{1}{4L}(n_2-n_1)^2+\frac{1}{2}(n_1+n_2)(s'_2-s_1)-\frac{L}{4}(s'_2-s_1)^2$. Therefore, we can combine the three inequalities in (12) into the following inequality:

$$\int_{s_1}^{s_2} n_\varepsilon(x_\varepsilon(s))ds \geq \frac{(n_1+n_2)}{2}(s'_2-s_1)-\frac{L}{2}(s'_2-s_1)^2 \quad (13)$$

where $s'_2 - s_1 = \|x_\varepsilon(s_2)-x_\varepsilon(s_1)\|$ $(= d(x_\varepsilon(s_1), x_\varepsilon(s_2)).)$

Now we compare this cost to the cost of an alternative path $y(s)$ that moves from $x_\varepsilon(s_1)$ to $x_\varepsilon(s_2)$ via 3 steps:

Step 1: Move up $\Gamma^{t_1}_{\lambda^*,\varepsilon}$. Specifically, $y(s) = \gamma(t_1)+(s-\varepsilon)\overrightarrow{n}(\gamma(t_1))$ where $0 \leq s \leq 2\varepsilon$.

Step 2: Move along $\Gamma^{-}_{\lambda^*,\varepsilon}$. Specifically, we start at $s = \sigma_1\,(=2\varepsilon)$ from $\gamma(t_1)+\varepsilon\overrightarrow{n}(\gamma(t_1))$ and move directly along $\gamma(t)+\varepsilon\overrightarrow{n}(\gamma(t))$ until $s=\sigma_2$, when we are at $\gamma(t_2)+\varepsilon\overrightarrow{n}(\gamma(t_2))$.

Step 3: Move down $\Gamma^{t_2}_{\lambda^*,\varepsilon}$. Specifically, $y(s) = \gamma(t_2)+(\sigma_2-s+\varepsilon)\overrightarrow{n}(\gamma(t_2))$ where $\sigma_2 \leq s \leq \sigma_2+2\varepsilon$.

Next we look for upper bounds on the cost of $n_\varepsilon(y(s))$. Since, by assumption, $n \leq B$, we can bound the cost for step 1 and step 3 by

$$\int_0^{2\varepsilon} n_\varepsilon(y(s))ds \leq 2B\varepsilon \text{ and } \int_{\sigma_2}^{\sigma_2+2\varepsilon} n_\varepsilon(y(s))ds \leq 2B\varepsilon.$$

We now perform an analysis for step 2 that is similar to our analysis for $x_\varepsilon(s)$. Define $N_1 = n(y(\sigma_1))$ and $N_2 = n(y(\sigma_2))$. From Lipschitz continuity (assumption (2b)) and the definition of $\Delta n_{\min}$, we know that

$$n_i - N_i \geq \Delta n_{\min} - 2\varepsilon L \text{ where } i=1,2. \quad (14)$$

Since $\left\|\frac{dy}{ds}\right\| = 1$ and $\Gamma^-_{\lambda^*,\varepsilon}$ borders the ε neighborhood of J, we know that

$$n_\varepsilon(y(s)) \le \begin{cases} N_1 + L(s-\sigma_1) & \text{if } \sigma_1 \le s \le (\sigma^* \wedge \sigma_2) \\ N_2 - L(s-\sigma_2) & \text{if } (\sigma_1 \vee \sigma^*) \le s \le \sigma_2 \end{cases}$$

where $\sigma^* \equiv \frac{1}{2}\left(\frac{n_1-n_2}{L} + \sigma_1 + \sigma_2\right)$, and so the cost can be bounded:

$$\int_{\sigma_1}^{\sigma_2} n_\varepsilon(y(s))ds \le \begin{cases} N_2(\sigma_2-\sigma_1) + \frac{L}{2}(\sigma_2-\sigma_1)^2 \;\; \text{if } \sigma^* \le \sigma_1 \\ N_1(\sigma^*-\sigma_1) + N_2(\sigma_2-\sigma^*) \\ \qquad + \frac{L}{2}\left((\sigma^*-\sigma_1)^2 + (\sigma_2-\sigma^*)^2\right) \text{if } \sigma_1 \le \sigma^* \le \sigma_2 \\ N_1(\sigma_2-\sigma_1) + \frac{L}{2}(\sigma_2-\sigma_1)^2 \text{ if } \sigma^* \ge \sigma_2. \end{cases} \tag{15}$$

As before, we note that (1) if $\sigma^* \le \sigma_1$, then $N_1 \ge N_2$, and therefore $N_2 \le \frac{N_1+N_2}{2}$, that (2) if $\sigma^* \ge \sigma_2$, then $N_2 \ge N_1$, and therefore $N_1 \le \frac{N_1+N_2}{2}$, and that (3) substituting the definition of σ^* into the right hand side of (15) for the case $\sigma_1 \le \sigma^* \le \sigma_2$, we see that it equals $-\frac{1}{4L}(N_2-N_1)^2 + \frac{1}{2}(N_1+N_2)(\sigma_2-\sigma_1) + \frac{L}{4}(\sigma_2-\sigma_1)^2$. Therefore, we can combine the three inequalities in (15) into the following inequality:

$$\int_{\sigma_1}^{\sigma_2} n_\varepsilon(y(s))ds \le \frac{(N_1+N_2)}{2}(\sigma_2-\sigma_1) + \frac{L}{2}(\sigma_2-\sigma_1)^2. \tag{16}$$

Now we look at the difference in cost between the two paths:

$$\begin{aligned} \Delta\text{cost} &\equiv \int_0^{\sigma_2+2\varepsilon} n_\varepsilon(y(s))ds - \int_{s_1}^{s_2} n_\varepsilon(x_\varepsilon(s))ds \\ &\le 4B\varepsilon + \int_{\sigma_1}^{\sigma_2} n_\varepsilon(y(s))ds - \int_{s_1}^{s_2} n_\varepsilon(x_\varepsilon(s))ds, \end{aligned}$$

apply (13) and (16):

$$\begin{aligned} \Delta\text{cost} &\le 4B\varepsilon + \frac{(N_1+N_2)}{2}(\sigma_2-\sigma_1) + \frac{L}{2}(\sigma_2-\sigma_1)^2 \\ &\quad - \frac{(n_1+n_2)}{2}(s_2'-s_1) + \frac{L}{2}(s_2'-s_1)^2, \\ &= 4B\varepsilon + \frac{(N_1-n_1)+(N_2-n_2)}{2}(s_2'-s_1) \\ &\quad + \frac{(N_1+N_2)}{2}\left[(\sigma_2-\sigma_1)-(s_2'-s_1)\right] + \frac{L}{2}\left[(s_2'-s_1)^2 + (\sigma_2-\sigma_1)^2\right], \end{aligned}$$

and apply (14) with the definition $\gamma = \frac{(\sigma_2 - \sigma_1)}{(s_2' - s_1)}$ which yields

$$\begin{aligned}\Delta\text{cost} \;\; &\leq \;\; 4B\varepsilon + \left[-\Delta n_{\min} + 2\varepsilon L + \frac{(N_1 + N_2)}{2}(\gamma - 1) \right] (s_2' - s_1) \\ &\quad + \frac{L}{2}(1 + \gamma^2)(s_2' - s_1)^2. \end{aligned} \tag{17}$$

Now we determine how large γ can be. γ is largest when Γ_{λ^*} is curved as far as possible to reduce Γ_ε^+ and extend Γ_ε^-. The radius of curvature, however, can be no smaller than R. This leads to the picture in Figure 3 where $\lambda^* \geq \lambda' \equiv t_2 - t_1$. From the geometry of this extreme case we have the following bound on γ:

$$\gamma \leq \frac{(R + \varepsilon)\theta}{(R - \varepsilon) 2 \sin\left(\frac{\theta}{2}\right)} \leq \frac{(R + \varepsilon)\theta}{(R - \varepsilon)\left(\theta - \frac{\theta^3}{24}\right)} = \frac{1 + \frac{\varepsilon}{R}}{\left(1 - \frac{\varepsilon}{R}\right)\left(1 - \frac{\lambda'^2}{24R^2}\right)}.$$

Now choose ε_0 and λ^* small enough that

$$\gamma \leq \frac{1 + \frac{\varepsilon}{R}}{\left(1 - \frac{\varepsilon}{R}\right)\left(1 - \frac{\lambda'^2}{24R^2}\right)} \leq \min\left[1 + \frac{\Delta n_{\min}}{2B}, \sqrt{3}\right]. \tag{18}$$

Applying (18) to (17) we have that

$$\Delta\text{cost} \leq 4B\varepsilon + \left[-\frac{\Delta n_{\min}}{2} + 2\varepsilon L \right] (s_2' - s_1) + \frac{L}{2}(1 + \gamma^2)(s_2' - s_1)^2,$$

and since $s_2' - s_1 = |x_\varepsilon(s_2) - x_\varepsilon(s_1)| \leq \lambda^* + 2\varepsilon$, (18) further implies that

$$\Delta\text{cost} \leq 4B\varepsilon + \left[-\frac{\Delta n_{\min}}{2} + 2\varepsilon L + 2L(\lambda^* + 2\varepsilon) \right] (s_2' - s_1).$$

Now choosing ε_0 and λ^* small enough that $2L(\lambda^* + 3\varepsilon) \leq \frac{\Delta n_{\min}}{4}$, we have

$$\Delta\text{cost} \leq 4B\varepsilon - \frac{\Delta n_{\min}}{4}(s_2' - s_1),$$

but, by definition, $x_\varepsilon(s)$ satisfies (8), therefore $-\epsilon(\varepsilon) \leq \Delta\text{cost}$, so if we choose $\epsilon(\varepsilon) \leq B\varepsilon$, then

$$s_2' - s_1 = |x_\varepsilon(s_2) - x_\varepsilon(s_1)| \leq K\varepsilon \quad \text{where } K \equiv \frac{20B}{\Delta n_{\min}}. \tag{19}$$

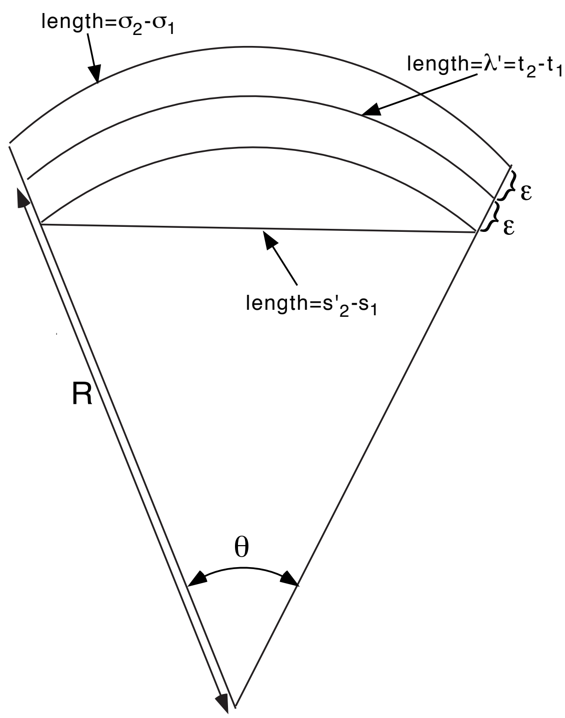

Figure 3: *The bound, R, on the radius of curvature leads to the above figure, which represents the largest possible value of* $\gamma = \frac{\sigma_2 - \sigma_1}{s_2 - s_1}$.

Since the entrance and exit points from $\Gamma_{\lambda^*,\varepsilon}$ on $\Gamma^+_{\lambda^*,\varepsilon}$ are, from (19), within $K\varepsilon$ of each other, we can now show that the path cannot stray more than $k\varepsilon$ from $\Gamma^+_{\lambda^*,\varepsilon}$.

Assume the contrary, that is, there is a point on the path corresponding to an $s_0 \in [s_1, s_2]$, $t_0 \in [0, \lambda^*]$, and $\alpha \geq k\varepsilon$ where

$$x_\varepsilon(s_0) = \gamma(t_0) - \alpha \overrightarrow{n}(\gamma(t_0)).$$

Restricting $\lambda^* < \frac{\pi R}{2}$ so that Γ_{λ^*} cannot bend more than a quarter circle, we have that

$$\begin{aligned} |x_\varepsilon(s_1) - x_\varepsilon(s_0)| &\geq (k-1)\varepsilon, \\ |x_\varepsilon(s_2) - x_\varepsilon(s_0)| &\geq (k-1)\varepsilon. \end{aligned}$$

Now restrict ε_0 so that $k\varepsilon_0 \leq \Delta$, and therefore

$$\begin{aligned} \int_{s_1}^{s_2} n_\varepsilon(x_\varepsilon(s))ds &= \int_{s_1}^{s_0} n_\varepsilon(x_\varepsilon(s))ds + \int_{s_0}^{s_2} n_\varepsilon(x_\varepsilon(s))ds \\ &\geq 2(k-1)n_0\varepsilon. \end{aligned} \tag{20}$$

Using (19) and the definition of $x_{\varepsilon,line}(s)$ in (9), we have that

$$\int_{s_1}^{s_2'} n_\varepsilon(x_{\varepsilon,line}(s))ds \leq B(s_2' - s_1) = B\,|x_\varepsilon(s_2) - x_\varepsilon(s_1)| \leq BK\varepsilon, \tag{21}$$

where we note that $x_{\varepsilon,line}(s)$ stays within Ω since the convex hull of $\Gamma^+_{\lambda^*,\varepsilon} \subset \Omega$. From (20) and (21) we obtain

$$-\epsilon(\varepsilon) \leq \int_{s_1}^{s_2'} n_\varepsilon(x_{\varepsilon,line}(s))ds - \int_{s_1}^{s_2} n_\varepsilon(x_\varepsilon(s))ds \leq BK\varepsilon - 2(k-1)n_0\varepsilon.$$

So, choose $\epsilon(\varepsilon) \leq \frac{BK}{4}\varepsilon$, which implies $2(k-1)n_0 \leq \frac{5}{4}BK$, and, therefore $k \leq \frac{5}{8}\frac{BK}{n_0} + 1$. Since $B \geq \Delta n_{\min}$, $B \geq n$, and, by the definition of K, $\frac{K\Delta n_{\min}}{20B} = 1$, we have $k \leq \frac{5}{8}\frac{BK}{n_0} + \frac{K\Delta n_{\min}}{20B} \leq \frac{5}{8}\frac{BK}{n_0} + \frac{K}{20} \leq \frac{5}{8}\frac{BK}{n_0} + \frac{1}{20}\frac{BK}{n_0}$, which contradicts the definition $k \equiv \frac{BK}{n_0} = \frac{20B^2}{n_0\Delta n_{\min}}$.

All that remains is to justify our initial assumption that $x_\varepsilon(s)$ does not intersect the ε neighborhood of J. Restrict ε_0 so that

$$4k\varepsilon_0 < \min_{\substack{i=1,2,\ldots,N \\ t_1,t_2\in[0,T_i] \\ |t_1-t_2|\geq\pi R}} d(\gamma_i(t_1), \gamma_i(t_2)) \text{ and} \tag{22}$$

$$4k\varepsilon_0 \leq \frac{\delta}{4}\sin(\theta). \tag{23}$$

Restriction (22) , which is just a modification of restriction (7) , prevents $x_\varepsilon(s)$ from coming within ε of the Γ_i that contains Γ_{λ^*}; restriction (23) combined with restriction (6) prevents $x_\varepsilon(s)$ from coming within ε of any Γ_j where $i \neq j$. With these restrictions on ε_0 in place, our assumption now holds and the proof of the lemma is complete. ■

Now we make explicit the relationship between the Γ_λ and the Γ_{λ^*} sections. We let Γ_λ be the middle section of Γ_{λ^*} where λ and λ^*, the lengths of Γ_λ and Γ_{λ^*}, are restricted by

$$\lambda \in \left(\frac{1}{2} \frac{\lambda^*}{2\beta+1}, \frac{\lambda^*}{2\beta+1} \right] \text{ where } \beta = 8\frac{B}{n_0} \tag{24}$$

(so there are sections of Γ_{λ^*} of arclength $\beta\lambda$ on both sides of Γ_λ). To ensure that $\Gamma_{\lambda^*} \subset J_{\delta^*}$ for any $\Gamma_\lambda \subset J_\delta$ we restrict λ^* so that $\frac{\lambda^*}{2\beta+1}\beta \leq \frac{\delta}{4}$. Further, we restrict λ^* so that $\frac{1}{2}\frac{\lambda^*}{2\beta+1}$ is less than the smallest non-zero arclength of the $\Gamma_{i,\delta}$, which allows us to describe any $\Gamma_{i,\delta}$ of non-zero arclength as the union of Γ_λ sections.

With our construction of the Γ_λ and Γ_{λ^*} sections, we can establish the next lemma which uses Lemma 4 to show that $x_\varepsilon(s)$ cannot start at and return to Γ_ε, if, between the start and the return, $x_\varepsilon(s)$ either strays far on the side of Γ where the n values are larger or $x_\varepsilon(s)$ gets too close (within ε_0) of discontinuities other than Γ_{λ^*}.

Lemma 5 *Let Γ_λ and Γ_{λ^*} be on Γ_i. For sufficiently small λ^* and ε_0, if $x_\varepsilon(s_1)$ and $x_\varepsilon(s_2) \in \Gamma_\varepsilon$, then for no $s_0 \in [s_1, s_2]$ can (1) $x_\varepsilon(s_0)$ be within ε_0 of $J - \Gamma_{\lambda^*}$ nor can (2) $x_\varepsilon(s_0) \in \Gamma_i^+$ unless there exists $t \in [0, \lambda^*]$ and $\alpha \in [0, k\varepsilon]$ where*

$$x_\varepsilon(s_0) = \gamma(t) - \alpha \overrightarrow{n}(\gamma(t)).$$

Proof. Restrict ε_0 to be less than $\frac{\lambda^*}{4(2\beta+1)}$ so that the distance between any two points in Γ_ε, which is bounded by $2\varepsilon_0 + \lambda$, is now less than 2λ. Therefore there exists a path $y(s)$ where $y(s_1) = x_\varepsilon(s_1)$, $y(s_2') = x_\varepsilon(s_2)$ and

$$\int_{s_1}^{s_2'} n_\varepsilon(y(s))ds \leq 2\lambda B. \tag{25}$$

We now show that for each of the three cases where the theorem is false, $s_2 - s_1$, which equals the distance travelled by the path x_ε, must be greater than $\frac{\beta\lambda}{2}$. Case 1) If $x_\varepsilon(s)$ goes from Γ_ε to a point within ε_0 of

$\Gamma_i - \Gamma_{\lambda^*}$ and back to Γ_ε then, for sufficiently small λ^* (specifically $\frac{\lambda^*}{(2\beta+1)}\beta \leq \frac{1}{2} \min_{\substack{i=1,2,\ldots,N \\ t_1,t_2\in[0,T_i] \\ |t_1-t_2|\geq \pi R}} d(\gamma_i(t_1), \gamma_i(t_2)))$ and small ε_0 (as already restricted in (22)), the distance travelled must be more than $\frac{\beta\lambda}{2}$. Case 2) From (6) and (23) we have that $\min_{\substack{i,j \\ i\neq j}} d(\Gamma_{i,\delta^*}, \Gamma_j) - 2\varepsilon_0 \geq \frac{\delta}{4}\sin\left(\frac{\theta}{2}\right)$, therefore restricting λ^* so that $\frac{\lambda^*}{2(2\beta+1)}\beta \leq \frac{\delta}{4}\sin\left(\frac{\theta}{2}\right)$, we have that any $x_\varepsilon(s)$ going from Γ_ε to a point within ε_0 of Γ_j $(i \neq j)$ and back to Γ_ε must travel a distance greater than $\frac{\beta\lambda}{2}$. Case 3) Since Γ_{λ^*} cannot bend more than a quarter circle, for sufficiently small ε_0 we have that all points in Γ_i^+ within distance $\frac{\beta\lambda}{2}$ of Γ_ε can be described by $\gamma(t) - \alpha\overrightarrow{n}(\gamma(t))$ for some $t \in [0, \lambda^*]$ and $\alpha \geq 0$. However, if for some s_0, $x(s_0) = \gamma(t) - \alpha\overrightarrow{n}(\gamma(t))$ for some $t \in [0, \lambda^*]$ and $\alpha \geq k\varepsilon$, then $x_\varepsilon(s)$ must, by Lemma 4, travel around one of the edges of $\Gamma^+_{\lambda^*,\varepsilon}$ which requires travelling a distance greater than $\frac{\beta\lambda}{2}$. For all three of these cases $n \geq n_0$ for at least the initial $\frac{\beta\lambda}{2}$ that must be travelled since $\frac{\beta\lambda}{2} \leq \frac{\delta}{4}\sin\left(\frac{\theta}{2}\right) \leq \Delta - \varepsilon_0$, therefore

$$\int_{s_1}^{s_2} n_\varepsilon(x_\varepsilon(s))ds \geq \frac{\beta\lambda n_0}{2}, \tag{26}$$

and, from (25) and (26), it follows that

$$-\epsilon(\varepsilon) \leq \int_{s_1}^{s_2'} n_\varepsilon(y(s))ds - \int_{s_1}^{s_2} n_\varepsilon(x_\varepsilon(s))ds \leq 2\lambda B - \frac{\beta\lambda n_0}{2}. \tag{27}$$

Now let $\epsilon(\varepsilon) < \lambda B$. Since $\beta = 8\frac{B}{n_0}$, (27) implies that $\lambda B \geq 2\lambda B$ which is clearly impossible, so, by contradiction, the lemma must be true. ■

Corollary 6 *Between the first time $x_\varepsilon(s)$ enters Γ_ε and the last time it exits Γ_ε, it can only be in region A or region B where*

$$\begin{aligned} \textit{region } A &\equiv \{x \in \Gamma_i^- : n_\varepsilon(x) = n^\varepsilon(x)\} \textit{ and} \\ \textit{region } B &\equiv \{x : x = \gamma(t) + \alpha\overrightarrow{n}(\gamma(t)) \textit{ where } \alpha \in [-k\varepsilon, \varepsilon] \textit{ and } t \in [0, \lambda^*]\}. \end{aligned}$$

Proof. The corollary follows directly from Theorem 5. ■

Next, define the first entrance of x_ε into Γ_ε and the last exit of x_ε from Γ_ε by

$$\begin{aligned} s_{in} &\equiv \min\{s : x_\varepsilon(s) \in \Gamma_\varepsilon\} \\ s_{out} &\equiv \max\{s : x_\varepsilon(s) \in \Gamma_\varepsilon\}. \end{aligned}$$

We are interested in comparing the n_ε based cost of $x_\varepsilon(s)$ between s_{in} and s_{out} with the n^ε based cost of a new path, which we will call $y(\cdot)$, that also starts at $x_\varepsilon(s_{in})$ and ends at $x_\varepsilon(s_{out})$. Before we specify the y path completely and compare costs in Lemma 8, we first establish Lemma 7, which compares the n_ε based cost of an x_ε path within region B with the n^ε based cost of a y path that runs along $\Gamma^-_{\lambda^*,\varepsilon}$ from the point on $\Gamma^-_{\lambda^*,\varepsilon}$ nearest the starting point of the x_ε path to the point on $\Gamma^-_{\lambda^*,\varepsilon}$ nearest the stopping point of the x_ε path.

Lemma 7 *: Let $x_\varepsilon(s) \in$ region B for all $s \in [s_1, s_2]$ and define $t_1, t_2 \in [0, \lambda^*]$ and $\alpha_1, \alpha_2 \in [-k\varepsilon, \varepsilon]$ so that*

$$\begin{aligned} x_\varepsilon(s_1) &= \gamma(t_1) + \alpha_1 \overrightarrow{n}(\gamma(t_1)) \text{ and} \\ x_\varepsilon(s_2) &= \gamma(t_2) + \alpha_2 \overrightarrow{n}(\gamma(t_2)). \end{aligned}$$

Let the path $y(s)$ be defined for $s \in [s_1, s_2']$ so that

$$\begin{aligned} y(s_1) &= \gamma(t_1) + \varepsilon \overrightarrow{n}(\gamma(t_1)), \\ y(s_2') &= \gamma(t_2) + \varepsilon \overrightarrow{n}(\gamma(t_2)), \end{aligned}$$

and $y(s)$ moves with speed $\left\|\frac{dy}{ds}\right\| = 1$ directly from $y(s_1)$ to $y(s_2')$ along the arc $\gamma(t) + \varepsilon \overrightarrow{n}(\gamma(t))$ where $t \in [t_1, t_2]$. Then

$$\int_{s_1}^{s_2} n_\varepsilon(x_\varepsilon(s))ds \geq \int_{s_1}^{s_2'} n^\varepsilon\left(y\left(s\right)\right)ds - C\varepsilon\left|t_2 - t_1\right|$$

where $C = (k+1)\left(\frac{B}{R} + L\right)$.

Proof. Define $n^-(t) = \lim_{\alpha \to 0^+} n\left(\gamma(t) + \alpha \overrightarrow{n}(\gamma(t))\right)$. From the Lipschitz continuity of n in assumption (2b) we have that $n^\varepsilon\left(\gamma(t) + \varepsilon \overrightarrow{n}(\gamma(t))\right) \leq n^-(t) + \varepsilon L$, and therefore

$$\int_{s_1}^{s_2'} n^\varepsilon\left(y\left(s\right)\right)ds \leq \left|\int_{t_1}^{t_2} (n^-(t) + \varepsilon L)\left|\frac{ds}{dt}\right| dt\right|.$$

Since R is a lower bound on the radius of curvature of the Γ_i, we have that $\left|\frac{ds}{dt}\right| \leq \frac{R+\varepsilon}{R}$, therefore

$$\int_{s_1}^{s_2'} n^\varepsilon\left(y\left(s\right)\right)ds \leq \left(1 + \frac{\varepsilon}{R}\right)\left|\int_{t_1}^{t_2} n^-(t)dt\right| + \left(\varepsilon L + \frac{\varepsilon^2 L}{R}\right)\left|t_2 - t_1\right|. \quad (28)$$

Similarly, $n_\varepsilon\left(\gamma(t)+\alpha\overrightarrow{n}(\gamma(t))\right)\geq n^-(t)-k\varepsilon L$ for $\alpha\in[-k\varepsilon,\varepsilon]$. Since $x_\varepsilon(s)$ is in region B, we have that 1) $\left|\frac{ds}{dt}\right|\geq\frac{R-k\varepsilon}{R}$ and 2) for each $s\in[s_1,s_2]$, there exists a t and $\alpha\in[-k\varepsilon,\varepsilon]$ such that $x_\varepsilon(s)=\gamma(t)+\alpha(s)\overrightarrow{n}(\gamma(t))$, therefore

$$\begin{aligned}\int_{s_1}^{s_2} n_\varepsilon(x_\varepsilon(s))ds &\geq \left|\int_{t_1}^{t_2}(n^-(t)-k\varepsilon L)\left(1-\frac{k\varepsilon}{R}\right)dt\right| \\ &\geq \left(1-\frac{k\varepsilon}{R}\right)\left|\int_{t_1}^{t_2} n^-(t)dt\right|-\left(k\varepsilon L-\frac{k^2\varepsilon^2 L}{R}\right)|t_2-t_1|.\end{aligned} \tag{29}$$

Subtracting (28) from (29) and using that $k\geq 1$, we have

$$\begin{aligned}\int_{s_1}^{s_2} n_\varepsilon(x_\varepsilon(s))ds &\geq \int_{s_1}^{s_2'} n^\varepsilon\left(y\left(s\right)\right)ds \\ &\quad -\varepsilon\left[\frac{k+1}{R}\left|\int_{t_1}^{t_2} n^-(t)dt\right|+(k+1)L\left|t_2-t_1\right|\right],\end{aligned}$$

and applying $n^-(t)\leq B$, we obtain our desired result. ■

With Corollary 6 and Lemma 7 in place, we can now proceed to Lemma 8 where we define a complete y path and compare its n^ε based cost with the n_ε based cost of x_ε between s_{in} and s_{out}.

Lemma 8 *There exists a path $y(\sigma)$ defined for $\sigma\in[\sigma_{in},\sigma_{out}]$ moving with speed $\left\|\frac{dy}{d\sigma}\right\|=1$ such that $x_\varepsilon(s_{in})=y(\sigma_{in})$, $x_\varepsilon(s_{out})=y(\sigma_{out})$, and*

$$\int_{s_{in}}^{s_{out}} n_\varepsilon(x_\varepsilon(s))ds\geq\int_{\sigma_{in}}^{\sigma_{out}} n^\varepsilon(y(\sigma))d\sigma-c\varepsilon \tag{30}$$

where the constant c is independent of ε.

Proof. From Corollary 6 we know that $x_\varepsilon(s)$ can only be in region A, the "nice" region where $n^\varepsilon=n_\varepsilon$ is guaranteed, or region B, where we must use more careful analysis. We begin by defining the entrances and exits of x_ε to and from region B within the interval $[s_{in},s_{out}]$. First set $s_{1,in}\equiv s_{in}$ and then inductively define

$$\begin{aligned}s_{i,out} &\equiv s_{out}\wedge\inf\{s>s_{i,in}:x_\varepsilon(s)\notin\text{region }B\} \\ s_{i,in} &\equiv \min\{s>s_{i-1,out}:x_\varepsilon(s)\in\text{region }B\}.\end{aligned}$$

Two cases are possible: there may be a finite integer I where $s_{I,out}=s_{out}$ or there is a countably infinite sequence of $s_{i,in}$ and $s_{i,out}$.

We first consider the case where the sequence is finite. Applying Corollary 6, we must have that

$$\begin{aligned} x_\varepsilon(s_{i,in}) &\in \Gamma^-_{\lambda^*,\varepsilon} \quad i = 2,3,...,I \\ x_\varepsilon(s_{i,out}) &\in \Gamma^-_{\lambda^*,\varepsilon} \quad i = 1,2,...,I-1. \end{aligned} \tag{31}$$

Therefore, defining $t_{i,in}, t_{i,out}, \alpha_{i,in}$, and $\alpha_{i,out}$ where $i = 1,2,...,I$ to satisfy

$$\begin{aligned} x_\varepsilon(s_{i,in}) &= \gamma(t_{i,in}) + \alpha_{i,in}\overrightarrow{n}(\gamma(t_{i,in})) \\ x_\varepsilon(s_{i,out}) &= \gamma(t_{i,out}) + \alpha_{i,out}\overrightarrow{n}(\gamma(t_{i,out})), \end{aligned}$$

we have from (31) that $\alpha_{i,in} = \alpha_{i,out} = \varepsilon$ except possibly for $\alpha_{1,in}$ and $\alpha_{I,out}$.

Let $y(\sigma)$ travel first along $\gamma(t_{1,in}) + \alpha\overrightarrow{n}(\gamma(t_{1,in}))$ as α varies from $\alpha_{1,in}$ to ε, and define $[\sigma_{in}, \sigma_{junction\,1}]$ as the range of values that σ takes along this path. Note that $y(\sigma_{in}) = x(s_{in})$ by construction and

$$\int_{\sigma_{in}}^{\sigma_{junction\,1}} n^\varepsilon(y(\sigma))d\sigma \le (k+1)B\varepsilon \tag{32}$$

since $\sigma_{junction\,1} - \sigma_{in} = \varepsilon - \alpha_{1,in} \le (k+1)\varepsilon$. Next, for $\sigma \in [\sigma_{junction\,1}, \sigma_{junction\,2}]$, let the path $y(\sigma)$ travel along $\gamma(t) + \varepsilon\overrightarrow{n}(\gamma(t))$ as t goes from $t_{1,in}$ to $t_{1,out}$. From Lemma 7, we have that

$$\int_{s_{1,in}}^{s_{1,out}} n_\varepsilon(x_\varepsilon(s))ds \ge \int_{\sigma_{junction\,1}}^{\sigma_{junction\,2}} n^\varepsilon(y(\sigma))d\sigma - C\varepsilon\,|t_{1,out} - t_{1,in}|\,. \tag{33}$$

If $I = 1$, then $\alpha_{1,out}$ may be something other than ε. Therefore, for $\sigma \in [\sigma_{junction\,2}, \sigma_{1,out}]$, let $y(\sigma)$ finally travel along $\gamma(t_{1,out}) + \alpha\overrightarrow{n}(\gamma(t_{1,out}))$ as α goes from ε to $\alpha_{1,out}$. Parallel to before, we note that $y(\sigma_{1,out}) = x_\varepsilon(s_{1,out})$ and

$$\int_{\sigma_{junction\,2}}^{\sigma_{1,out}} n^\varepsilon(y(\sigma))d\sigma \le (k+1)B\varepsilon. \tag{34}$$

Combining (32), (33), and (34), we have that

$$\int_{s_{in}}^{s_{1,out}} n_\varepsilon(x_\varepsilon(s))ds \ge \int_{\sigma_{in}}^{\sigma_{1,out}} n^\varepsilon\left(y\left(\sigma\right)\right)d\sigma - C\varepsilon\,|t_{1,out} - t_{1,in}| - 2(k+1)B\varepsilon. \tag{35}$$

We note that any $x_\varepsilon(s)$ that crosses itself has a corresponding lower cost path that excludes the section of the path that loops back to itself. Therefore, without loss of generality, we can strictly consider $x_\varepsilon(s)$ that do not intersect themselves. Now define the interval $[[a,b]] \equiv [\min\{a,b\}, \max\{a,b\}]$. Since $x_\varepsilon(s)$ cannot cross itself, for $i,j \in 2,3,...,I-1$,

$$[[t_{i,in}, t_{i,out}]] \cap [[t_{j,in}, t_{j,out}]] = [[t_{i,in}, t_{i,out}]], [[t_{j,in}, t_{j,out}]], \text{ or } \emptyset. \tag{36}$$

Take the sets $\{[[t_{2,in}, t_{2,out}]], ..., [[t_{N-1,in}, t_{N-1,out}]]\}$ and from (36) we see that we can form $M(< I-2)$ equivalence classes from these sets such that the union of all sets within an equivalence class equals one of the sets in the class and the intersection of any two sets from different classes is empty.

Now start with the class that contains $[[t_{2,in}, t_{2,out}]]$. Define $\mu_1 = 2$ and let ν_1 be the largest integer where $[[t_{\nu_1,in}, t_{\nu_1,out}]]$ is in the class. We are interested in the behavior of $x_\varepsilon(s)$ between $s = s_{\mu_1,in}$ and $s = s_{\nu_1,out}$. Let $\rho_1 \in [\mu_1, \nu_1]$ refer to the set $[[t_{\rho_1,in}, t_{\rho_1,out}]]$ that contains the other sets in the equivalence class. There must exist $s_1', s_2' \in [s_{\rho_1,in}, s_{\rho_1,out}]$ where

$$\begin{aligned} x_\varepsilon(s_1') &= \gamma(t_{\mu_1,in}) + \alpha \overrightarrow{n}\left(\gamma(t_{\mu_1,in})\right) \\ x_\varepsilon(s_2') &= \gamma(t_{\nu_1,out}) + \alpha \overrightarrow{n}\left(\gamma(t_{\nu_1,out})\right) \end{aligned}$$

and $\forall s \in [s_1', s_2']$, $x(s) \in$ region B. Clearly

$$\int_{s_{\mu_1,in}}^{s_{\nu_1,out}} n_\varepsilon(x_\varepsilon(s))ds \geq \int_{s_1'}^{s_2'} n_\varepsilon(x_\varepsilon(s))ds \tag{37}$$

since $s_{\mu_1,in} \leq s_1'$ and $s_2' \leq s_{\nu_1,out}$. Now we apply Lemma 7 to the right hand side of (37), and we have that

$$\int_{s_{\mu_1,in}}^{s_{\nu_1,out}} n_\varepsilon(x_\varepsilon(s))ds \geq \int_{\sigma_1}^{\hat{\sigma}_1} n^\varepsilon(y(\sigma))d\sigma - C\varepsilon \left|t_{\nu_1,out} - t_{\mu_1,in}\right|$$

where $y(s)$ moves along $\Gamma^-_{\lambda^*,\varepsilon}$, $y(\sigma_1) = x_\varepsilon(s_{\mu_1,in})$, and $y(\hat{\sigma}_1) = x_\varepsilon(s_{\nu_1,out})$.

Now define $\mu_2 = \nu_1 + 1$ and define ν_2 to be the largest number where $[[t_{\nu_2,in,} t_{\nu_2,out}]]$ is in the same equivalence class as $[[t_{\mu_2,in,} t_{\mu_2,out}]]$. Repeat the process for a total of p times where p is defined by $\nu_p = I-1$. From our definition of equivalence class, we see that for $i,j \in \{1,2,...,p\}$, $[[t_{\mu_i,in,} t_{\nu_i,out}]] \cap [[t_{\mu_j,in,} t_{\nu_j,out}]] = \emptyset$ and therefore

$$\begin{aligned} \sum_{i=1}^{p} \int_{s_{\mu_i,in}}^{s_{\nu_i,out}} n_\varepsilon(x_\varepsilon(s))ds &\geq \sum_{i=1}^{p} \int_{\sigma_i}^{\hat{\sigma}_i} n^\varepsilon(y(\sigma))d\sigma - C\varepsilon \sum_{i=1}^{p} \left|t_{\nu_i,out} - t_{\mu_i,in}\right| \quad (38) \\ &\geq \sum_{i=1}^{p} \int_{\sigma_i}^{\hat{\sigma}_i} n^\varepsilon(y(\sigma))d\sigma - C\lambda^*\varepsilon \end{aligned}$$

where $y(\sigma_i) = y(s_{\mu_i,in})$ and $y(\hat{\sigma}_i) = y(s_{\nu_i,out})$.

Now define $\sigma_{I,in}$ and σ_{out} so that $y(\sigma_{I,in}) = x_\varepsilon(s_{I,in})$ and $y(\sigma_{out}) = x_\varepsilon(s_{out})$ where $y(\sigma)$ travels first along $\gamma(t) + \varepsilon\overrightarrow{n}(\gamma(t))$ as t varies from $t_{I,in}$ to $t_{I,out}$ and then along $\gamma(t_{I,out}) + \alpha\overrightarrow{n}(\gamma(t_{I,out}))$ as α varies from ε to $\alpha_{I,out}$. From the same logic used on the $[\sigma_{1,in}, \sigma_{1,out}]$ interval to establish (35), we have that

$$\int_{s_{I,in}}^{s_{out}} n_\varepsilon(x_\varepsilon(s))ds \geq \int_{\sigma_{I,in}}^{\sigma_{out}} n^\varepsilon(y(\sigma))d\sigma - C\varepsilon\,|t_{I,out} - t_{I,in}| - (k+1)B\varepsilon. \quad (39)$$

Within region A, $n^\varepsilon = n_\varepsilon$, so when x_ε is in region A (i.e., when it is not in region B) we define the path y to be the same as x_ε so that the same cost is incurred; that is,

$$\begin{aligned}
\int_{s_{1,out}}^{s_{2,in}} n_\varepsilon(x_\varepsilon(s))ds &= \int_{\sigma_{1,out}}^{\sigma_1} n^\varepsilon(y(\sigma))d\sigma && (40)\\
\int_{s_{\nu_i,out}}^{s_{\mu_{i+1},in}} n_\varepsilon(x_\varepsilon(s))ds &= \int_{\hat{\sigma}_i}^{\sigma_{i+1}} n^\varepsilon(y(\sigma))d\sigma \qquad i = 1, 2, ..., p-1 &&\\
\int_{s_{I-1,out}}^{s_{I,in}} n_\varepsilon(x_\varepsilon(s))ds &= \int_{\hat{\sigma}_p}^{\sigma_{I,in}} n^\varepsilon(y(\sigma))d\sigma. &&
\end{aligned}$$

Adding (35), (38), (39), and (40) would establish the lemma for the case where $s_{i,in}$ and $s_{i,out}$ are finite sequences; we are now ready to return to the case where the $s_{i,in}$ and $s_{i,out}$ are infinite sequences. Since the cost associated with $x_\varepsilon(s)$ cannot be infinite, it follows that $\sum_{i=2}^{\infty} \bar{d}(x_\varepsilon(s_{i,in}), x_\varepsilon(s_{i,out})) < \infty$ where $\bar{d}(a, b)$ for $a, b \in \Gamma^-_{\lambda^*,\varepsilon}$ is defined to be the distance along $\Gamma^-_{\lambda^*,\varepsilon}$ between a and b. (We note that "$\sum_{i=2}^{\infty}$" is meant here to denote summing over all i except both $i = 1$, which corresponds to s_{in}, *and* the "final" i, which corresponds to s_{out}. This distinction is necessary since $x_\varepsilon(s_{in})$ and $x_\varepsilon(s_{out})$ may not be on $\Gamma^-_{\lambda^*,\varepsilon}$.) Therefore there exists a subsequence $\{i_j\}_{j=1}^{I}$ (where I depends on ε) such that $s_{i_1} = s_{in}$, $s_{i_I} = s_{out}$, and

$$\sum_{j=2}^{I-1} \bar{d}(x_\varepsilon(s_{i_j,in}), x_\varepsilon(s_{i_j,out})) + \varepsilon > \sum_{i=2}^{\infty} \bar{d}(x_\varepsilon(s_{i,in}), x_\varepsilon(s_{i,out})). \quad (41)$$

Now we relabel $s_j = s_{i_j}$ $j = 1, 2, ..., I$ and repeat the finite case analysis until we reach the analysis of region A, which we alter in the following manner:

In the regions $[s_{1,out}, s_{2,in}]$, $[s_{\nu_i,out}, s_{\mu_{i+1},in}]$ $i = 1, 2, ..., p-1$, and $[s_{I-1,out}, s_{I,in}]$, we can no longer guarantee that we are in region A. However, we can take advantage of (41) to minimize the effect of the time in region B. As before, we let the path y be the same as x_ε when x_ε is in region A. As soon as x_ε enters region B, however, y moves along $\Gamma^-_{\lambda^*,\varepsilon}$ to the next location where x_ε will leave region B and enter region A again. Even assuming the extreme case where x_ε has no cost when it is in region B and y has the maximum cost, B, when it moves along $\Gamma^-_{\lambda^*,\varepsilon}$, we still have from (41) the following analogue of (40):

$$\begin{aligned}
\int_{s_{1,out}}^{s_{2,in}} n_\varepsilon(x_\varepsilon(s))ds &\geq \int_{\sigma_{1,out}}^{\sigma_{2,in}} n^\varepsilon(y(\sigma))d\sigma - Bf_0\varepsilon \qquad (42)\\
\int_{s_{\nu_i,out}}^{s_{\mu_{i+1},in}} n_\varepsilon(x_\varepsilon(s))ds &\geq \int_{\hat{\sigma}_i}^{\sigma_{i+1}} n^\varepsilon(y(\sigma))d\sigma \ - Bf_i\varepsilon \qquad i=1,2,...,p-1\\
\int_{s_{I-1,out}}^{s_{I,in}} n_\varepsilon(x_\varepsilon(s))ds &\geq \int_{\hat{\sigma}_p}^{\sigma_{I,in}} n^\varepsilon(y(\sigma))d\sigma - Bf_p\varepsilon
\end{aligned}$$

where $\sum_{i=0}^{p} f_i < 1$. Combining (35), (38), (39), and (42) we have that

$$\begin{aligned}
&\int_{s_{in}}^{s_{out}} n_\varepsilon(x_\varepsilon(s))ds \geq \int_{\sigma_{in}}^{\sigma_{out}} n^\varepsilon(y(\sigma))d\sigma \\
&-\varepsilon\left[\begin{array}{c} C\left(|t_{1,out}-t_{1,in}|+|t_{I,out}-t_{I,in}|+\lambda^*\right) \\ +3(k+1)B+\sum_{i=0}^{p} f_iB \end{array}\right] \\
\geq\ &\int_{\sigma_{in}}^{\sigma_{out}} n^\varepsilon(y(\sigma))d\sigma - \varepsilon\left[3C\lambda^*+(3k+4)B\right]
\end{aligned}$$

where $x_\varepsilon(s_{in}) = y(\sigma_{in})$ and $x_\varepsilon(s_{out}) = y(\sigma_{out})$. Since C, β, λ^*, k, and B are not dependent upon ε, the lemma is established where $c \equiv 3C\lambda^* + (3k+4)B$. ■

Finally, in Lemma 9, we determine a cost comparison similar to Lemma 8, but applied to an arbitrary $B_\delta(p)$ region, as opposed to an arbitrary Γ_ε region. Echoing the Γ_ε analysis, we first define $s_{in,\delta}$ to be the first time x_ε enters $B_\delta(p)$ and $s_{out,\delta}$ to be the last time x_ε exits $B_\delta(p)$; specifically,

$$\begin{aligned}
s_{in,\delta} &= \inf\{s : x_\varepsilon(s) \in B_\delta(p)\}\\
s_{out,\delta} &= \sup\{s : x_\varepsilon(s) \in B_\delta(p)\}.
\end{aligned}$$

Now we proceed to the cost comparison:

Lemma 9 *There exists a path $y(\sigma)$ defined for $\sigma \in [\sigma_{in,\delta}, \sigma_{out,\delta}]$ moving with speed $\left\|\frac{dy}{d\sigma}\right\| = 1$ such that $x_\varepsilon(s_{in,\delta}) = y(\sigma_{in,\delta})$, $x_\varepsilon(s_{out,\delta}) = y(\sigma_{out,\delta})$, and*

$$\int_{s_{in,\delta}}^{s_{out,\delta}} n_\varepsilon(x_\varepsilon(s))ds \geq \int_{\sigma_{in,\delta}}^{\sigma_{out,\delta}} n^\varepsilon\left(y\left(\sigma\right)\right)d\sigma - B\kappa\delta. \qquad (43)$$

Proof. Define $y(\sigma)$ to be the path that starts at $x_\varepsilon(s_{in,\delta})$ when $\sigma = \sigma_{in,\delta}$, then moves as directly as possible to $x_\varepsilon(s_{out,\delta})$, which it reaches at $\sigma = \sigma_{out,\delta}$. Since, by assumption (3f) and the restriction from the end of subsection 3.2 that $\delta_0 \le \rho$, we have that $\sigma_{out,\delta} - \sigma_{in,\delta} \le \kappa\delta$, and so it follows that

$$\int_{s_{in,\delta}}^{s_{out,\delta}} n_\varepsilon(x_\varepsilon(s))ds \ge 0 \ge \int_{\sigma_{in,\delta}}^{\sigma_{out,\delta}} n^\varepsilon\left(y\left(\sigma\right)\right)d\sigma - B\kappa\delta,$$

and the lemma is established. ■

3.4 Establishing the main theorem

Since the region of Ω where $n^\varepsilon \neq n_\varepsilon$ is a subset of the union of Γ_ε and $B_\delta(p)$ regions, we can piece together the local results from Lemmas 8 and 9 to compare the n_ε cost of the *entire* $x_\varepsilon(s)$ path with the n^ε cost of a $y(\sigma)$ path. This leads to the sought after lower bound on u_ε in terms of u^ε, which we combine with the fact that $u_\varepsilon \le u^\varepsilon$ from monotonicity to establish our main theorem:

Theorem 10 $\lim_{\varepsilon\to 0} u_\varepsilon(X) = \lim_{\varepsilon\to 0} u^\varepsilon(X) \qquad \forall X \in \Omega.$

Proof. By construction, we can divide each $\Gamma_{i,\delta}$ into $n_i \equiv \left\lceil \frac{T_i}{\frac{\lambda^*}{2\beta+1}} \right\rceil$ continuous pieces, which we label $\Gamma_{ij,}$ (for $i = 1, 2, ..., N$; $j = 1, 2, ..., n_i$) where λ_{ij}, the arclength of Γ_{ij}, always conforms to the length restrictions on λ in (24). Also, define m ($\le 2N$) to be the number of distinct endpoints, p. Finally, defining $\Gamma_{ij,\varepsilon} \equiv \cup_{\alpha\in[-\varepsilon,\varepsilon]} \cup_{t:\gamma_i(t)\in\Gamma_{ij}} \left[\gamma(t) + \alpha\overrightarrow{n}(\gamma(t))\right]$, we divide Ω into three distinct regions:

$$\begin{aligned}
\text{Region 1} &\equiv \cup_{i=1}^m B_\delta(p_i) \cap \Omega, \\
\text{Region 2} &\equiv \cup_{i=1}^N \cup_{j=1}^{n_i} \Gamma_{ij,\varepsilon} \ - \text{Region 1}, \\
\text{Region 3} &\equiv \{\Omega - (\text{Region 1} \ \cup \ \text{Region 2})\}.
\end{aligned}$$

Note that, by construction, $n^\varepsilon(X) = n_\varepsilon(X)$ at any $X \in$ Region 3.

As in the previous subsection, $x_\varepsilon(s)$ will represent a path that (almost) minimizes the cost for $n_\varepsilon(x(s))$; i.e.,

$$u_\varepsilon(X) - \left[\int_0^S n_\varepsilon(x_\varepsilon(s))ds + g(x(S))\right] \ge -\epsilon(\varepsilon) \tag{44}$$

where $x_\varepsilon(0) = X$. Now we use the following algorithm to partition $[0, S]$ into segments defined by the region in which $x_\varepsilon(s)$ is located:

a) Step 0: Initialize $i = 0, j = 0, k = 0$, and $s = 0$. If $X \in$ Region 1, go to step 1; if $X \in$ Region 2, go to step 2; if $X \in$ Region 3, go to step 3.

b) Step 1: Add 1 to i. Define $s^1_{i,in} \equiv s$. Let p_l denote the (unique, by construction) point such that $x_\varepsilon(s) \in B_\delta(p_l)$. Define $s^1_{i,out} \equiv \sup\{s : x_\varepsilon(s) \in B_\delta(p_l)\}$. Update $s \equiv s^1_{i,out}$. If $s^1_{i,out} = S$ then stop; otherwise go to step 4.

c) Step 2: Add 1 to j. Define $s^2_{j,in} \equiv s$. Let l, m denote a region $\Gamma_{lm,\varepsilon}$ where $x_\varepsilon(s) \in \Gamma_{lm,\varepsilon}$. Define $s^2_{j,out} \equiv \sup\{s : x_\varepsilon(s) \in \Gamma_{lm,\varepsilon}\}$. Update $s \equiv s^2_{j,out}$. If $s^2_{j,out} = S$ then stop; otherwise go to step 4.

d) Step 3: Add 1 to k. Define $s^3_{k,in} \equiv s$. Define $s^3_{k,out} \equiv \sup\{s > s^3_{k,in} : \forall s' \in (s^3_{k,in}, s), x_\varepsilon(s') \in \text{region } 3\}$. Update $s \equiv s^3_{k,out}$. If $s^3_{k,out} = S$ then stop; otherwise go to step 4.

e) Step 4: If there exists $\varepsilon > 0$ such that $\forall \varepsilon' \in (0, \varepsilon)$, $x(s+\varepsilon') \in$ region 1, then go to step 1; if there exists $\varepsilon > 0$ such that $\forall \varepsilon' \in (0, \varepsilon)$, $x(s+\varepsilon') \in$ region 2, then go to step 2; otherwise there must exist $\varepsilon > 0$ such that $\forall \varepsilon' \in (0, \varepsilon)$, $x(s + \varepsilon') \in$ region 3, so go to step 3.

Now define m_δ to be the value of i when the algorithm ends; m_ε to be the value of j; and m_0 to be the value of k. Note that $m_\delta \leq m$, $m_\varepsilon \leq \sum_{i=1}^N n_i$, and

$$[0, S) = \cup_{i=1}^{m_\delta}[s^1_{i,in}, s^1_{i,out}) \cup_{j=1}^{m_\varepsilon} [s^2_{j,in}, s^2_{j,out}) \cup_{k=1}^{m_0} [s^3_{k,in}, s^3_{k,out}) \tag{45}$$

where each of the intervals on the right hand side of (45) are distinct.

Now we use Lemmas 8 and 9 to construct the entire path $y(\sigma)$ and compare the n^ε cost of $y(\sigma)$ with the n_ε cost of $x_\varepsilon(s)$:

For $i = 1, 2, ..., m_\delta$, define $y(\sigma^1_{i,in}) = x_\varepsilon(s^1_{i,in})$ and $y(\sigma^1_{i,out}) = x_\varepsilon(s^1_{i,out})$. Define the path $y(\sigma)$ for $\sigma \in (\sigma^1_{i,in}, \sigma^1_{i,out})$ as specified in Lemma 9, and therefore, from the conclusion to Lemma 9, we have that

$$\int_{s^1_{i,in}}^{s^1_{i,out}} n_\varepsilon(x_\varepsilon(s))ds \geq \int_{\sigma^1_{i,in}}^{\sigma^1_{i,out}} n^\varepsilon\left(y\left(\sigma\right)\right) d\sigma - B\kappa\delta \tag{46}$$

where $i = 1, 2, ..., m_\delta$.

For $j = 1, 2, ..., m_\varepsilon$, define $y(\sigma^2_{j,in}) = x_\varepsilon(s^2_{j,in})$ and $y(\sigma^2_{j,out}) = x_\varepsilon(s^2_{j,out})$. Define $y(\sigma)$ for $\sigma \in (\sigma^2_{j,in}, \sigma^2_{j,out})$ as specified in Lemma

8, and therefore, from the conclusion to Lemma 8, we have that

$$\int_{s^2_{j,in}}^{s^2_{j,out}} n_\varepsilon(x_\varepsilon(s))ds \geq \int_{\sigma^2_{j,in}}^{\sigma^2_{j,out}} n^\varepsilon\left(y\left(\sigma\right)\right)d\sigma - c\varepsilon \tag{47}$$

where $j = 1, 2, ..., m_\varepsilon$.

Finally, for $k = 1, 2, ..., m_0$, define $y(\sigma^3_{k,in}) = x_\varepsilon(s^3_{k,in})$ and $y(\sigma^3_{k,out}) = x_\varepsilon(s^3_{k,out})$. For all $\sigma \in (0, s^3_{k,out} - s^3_{k,in})$ define $y(\sigma^3_{k,in} + \sigma) = x_\varepsilon(s^3_{k,in} + \sigma)$. That is, the paths are the same. Since $n^\varepsilon = n_\varepsilon$ in this region, we have that

$$\int_{s^3_{k,in}}^{s^3_{k,out}} n_\varepsilon(x_\varepsilon(s))ds = \int_{\sigma^3_{k,in}}^{\sigma^3_{k,out}} n^\varepsilon\left(y\left(\sigma\right)\right)d\sigma \tag{48}$$

where $k = 1, 2, ..., m_0$.

With our construction of $y(\sigma)$ in place, we now combine (45), (46), (47), and (48) to obtain

$$\begin{aligned}\int_0^S n_\varepsilon(x_\varepsilon(s))ds &= \sum_{i=1}^{m_\delta}\int_{s^1_{i,in}}^{s^1_{i,out}} n_\varepsilon(x_\varepsilon(s))ds \qquad (49)\\ &\quad + \sum_{j=1}^{m_\varepsilon}\int_{s^2_{j,in}}^{s^2_{j,out}} n_\varepsilon(x_\varepsilon(s))ds + \sum_{k=1}^{m_0}\int_{s^3_{k,in}}^{s^3_{k,out}} n_\varepsilon(x_\varepsilon(s))ds\\ &\geq \sum_{i=1}^{m_\delta}\int_{\sigma^1_{i,in}}^{\sigma^1_{i,out}} n^\varepsilon\left(y\left(\sigma\right)\right)d\sigma + \int_{\sigma^2_{j,in}}^{\sigma^2_{j,out}} n^\varepsilon\left(y\left(\sigma\right)\right)d\sigma\\ &\quad + \sum_{k=1}^{m_0}\int_{\sigma^3_{k,in}}^{\sigma^3_{k,out}} n^\varepsilon\left(y\left(\sigma\right)\right)d\sigma \; - B\kappa m\delta - \left(\sum_{i=1}^{N} n_i\right)c\varepsilon\\ &\geq \int_0^\Sigma n^\varepsilon\left(y\left(\sigma\right)\right)d\sigma - B\kappa m\delta - C\varepsilon\end{aligned}$$

where $\Sigma = \max\{\sigma^1_{m_\delta,out}, \sigma^2_{m_\varepsilon,out}, \sigma^3_{m_0,out}\}$ and C is a constant that may depend on δ but not on ε.

Since $y(0) = X$, $y(\Sigma) = x_\varepsilon(S)$, and $y(\sigma)$ is a continuous path, we have that

$$u^\varepsilon(X) \leq \int_0^\Sigma n^\varepsilon\left(y\left(\sigma\right)\right)d\sigma + g(y(\Sigma)). \tag{50}$$

Combining this with (44) and (49) yields

$$u_\varepsilon(X) \geq u^\varepsilon(X) - B\kappa m\delta - C\varepsilon - \epsilon(\varepsilon). \tag{51}$$

Now choose $\epsilon(\varepsilon)$ such that $0 < \epsilon(\varepsilon) \leq \varepsilon$ and let $\varepsilon \to 0$ in (50) to obtain

$$\lim_{\varepsilon \to 0} u_\varepsilon(X) \geq \lim_{\varepsilon \to 0} u^\varepsilon(X) - B\kappa m\delta \tag{52}$$

where $\lim_{\varepsilon \to 0} u_\varepsilon$ and $\lim_{\varepsilon \to 0} u^\varepsilon$ exist by Corollary 3. But δ is arbitrarily small, therefore

$$\lim_{\varepsilon \to 0} u_\varepsilon(X) \geq \lim_{\varepsilon \to 0} u^\varepsilon(X). \tag{53}$$

Further, from Corollary 2, we know that $u_\varepsilon(X) \leq u^\varepsilon(X)$, therefore

$$\lim_{\varepsilon \to 0} u_\varepsilon(X) \leq \lim_{\varepsilon \to 0} u^\varepsilon(X), \tag{54}$$

and we combine (53) and (54) to obtain our final result:

$$\lim_{\varepsilon \to 0} u_\varepsilon(X) = \lim_{\varepsilon \to 0} u^\varepsilon(X) \qquad \forall X \in \Omega. \blacksquare$$

4 Continuity and control formulation of u

We can now define $u(X) \equiv \lim_{\varepsilon \to 0} u^\varepsilon(X) = \lim_{\varepsilon \to 0} u_\varepsilon(X)$ for all $X \in \Omega$ to be the (unique) solution to (1). To establish the Lipschitz continuity of u, we first show that the u^ε (and also the u_ε) are equilipschitz. Since $n(x) \leq B$ implies that $\|Du\| \leq B \ \forall X \in \Omega$ where (1) is satisfied in the classical sense, it is not particularly surprising that we can extend this Lipschitz continuity to the entire region Ω in the following theorem:

Theorem 11 *u^ε and u_ε are equilipschitz continuous in Ω with Lipschitz constant B.*

Proof. Let n^* be any specific n^ε or n_ε and u^* be the corresponding solution. We note that $n^* \leq B$. Consider any $X_0 \in \Omega$. By definition

$$u^*(X_0) = \inf_{\alpha(\cdot) \in A} \left[\int_0^S n^*(x(s))ds + g(x(S)) \right],$$

where $\dot{x}(s) = \alpha(s)$ and $x(0) = X_0$. Now choose $\delta > 0$ such that $B_\delta(X_0) \subset \Omega$ and choose any $X_1 \in B_\delta(X_0)$. If we only consider paths that first travel directly from X_0 to X_1, our cost is higher:

$$u^*(X_0) \leq \int_0^t n^*(x(s))ds + \inf_{\alpha(\cdot) \in \widetilde{A}} \left[\int_t^S n^*(x(s))ds + g(x(S)) \right]$$

where $t = \|X_1 - X_0\|$, $\widetilde{A} = \{\alpha : [t, S] \to S^1 \mid \alpha(\cdot) \text{ is measurable}\}$ and the dynamics are

$$\dot{x}(s) = \begin{cases} \frac{X_1 - X_0}{|X_1 - X_0|} \text{ if } s \in [0, t) \\ \alpha(s) \text{ if } s \in [t, S] \end{cases}$$

where $x(0) = X_0$ and $x(S) \in \partial\Omega$. Since $x(t) = X_1$, we have, by the definition of $u(X)$, that

$$u(X_0) \leq \int_0^t n^*(x(s))ds + u(X_1). \tag{55}$$

Repeating the above argument with the roles of X_0 and X_1 switched yields

$$u(X_0) \leq \int_0^t n^*(x(s))ds + u(X_1). \tag{56}$$

Therefore, from (55) and (56),

$$\begin{aligned} |u(X_1) - u(X_0)| \leq \int_0^t n^*(x(s))ds \\ \leq \quad & Bt \\ = \quad & B\|X_1 - X_0\|. \blacksquare \end{aligned}$$

Since the u^ε are equilipschitz, it follows from functional analysis that the u^ε must converge uniformly on Ω to u, which must also be Lipschitz continuous with Lipschitz constant B, and, by Rademacher's Theorem, Du exists almost everywhere in Ω. Further, if the u^ε are all continuous on $\overline{\Omega}$, then u is continuous and Lipschitz on $\overline{\Omega}$. (We note that it is not uncommon for the u_ε to fail to be continuous at the boundary of Ω even when u is continuous since $n_\varepsilon \leq n$ means that u_ε cannot "stretch" as far as u can; on the other hand, since $n^\varepsilon \geq n$, u^ε can "stretch" more than u and so, if u is continuous, the u^ε will be as well.)

Finally, we conclude with a control representation of u. Since n is only defined on $\Omega - J$, we extend the domain of n to Ω by defining $n^{ext}(X)$ to equal $n(X)$ for $X \in \Omega - J$ and, for $X \in J \cap \Omega$, $n^{ext}(X)$ can take any value in the interval $\left[\liminf_{Y \to X} n(Y), \limsup_{Y \to X} n(Y)\right]$. Since $n_\varepsilon(X) \leq n^{ext}(X) \leq n^\varepsilon(X)$ for all $\varepsilon > 0$ and $X \in \Omega$, it follows directly from the definition of u and Theorems 1 and 10 that u can be represented by

$$u(X) = \inf_{\alpha(\cdot) \in A} \left[\int_0^S n^{ext}(x(s))ds + g(x(S))\right]$$

with dynamics

$$\dot{x}(s) = \alpha(s) \text{ where } x(0) = X,\ x(S) \in \partial\Omega, \text{ and } \|\alpha(s)\| = 1\ \forall s \in [0, S].$$

Acknowledgments: I would like to acknowledge the financial support of the National Science Foundation under Grant No. DMS-9704864.

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