

## An Option to Reduce Transaction Costs\*

Jonathan Goodman<sup>†</sup> and Daniel N. Ostrov<sup>‡</sup>

**Abstract.** For small transaction costs, we determine the leading order optimal dynamic trading strategy of a portfolio of stock, cash, and options. Except for the transaction costs, our market assumptions are those of Black, Scholes, and Merton. Without transaction costs, the option is redundant in the portfolio. With transaction costs, however, we show that adding the option to the portfolio can significantly reduce overall trading costs compared to optimal strategies that use only stock and cash. The analysis is based on an asymptotic expansion with three scales: macroscopic, mesoscopic, and microscopic. The macroscopic analysis is Merton's optimal investment problem. Within a plane defined by the amount of stock and options held, the macroscopic analysis yields a *Merton line* of optimal portfolios. We show that there is a particular *magic point* on the Merton line that minimizes expensive stochastic movement away from the Merton line. The mesoscopic scale governs less expensive deviations of the portfolio away from the magic point but along the Merton line. The microscopic scale governs the more expensive deviations of the portfolio away from the magic point, transverse to the Merton line. The resulting strategy is related to commonly used Delta and Gamma hedging strategies, but our scale analysis implies that some rebalancings are much more effective than others. We do not give rigorous mathematical proofs, only arguments of formal applied mathematics.

**Key words.** transaction costs, Merton strategy, asymptotics, optimal rebalancing

**AMS subject classifications.** 91G10, 91G80, 49K10, 41A60

**DOI.** 10.1137/100798053

**1. Introduction.** Many trading strategies used in modern financial practice make use of options as well as underlying assets. A mathematical analysis of such strategies must use a market model that goes beyond the highly idealized model of Black, Scholes, and Merton. Options may be replicated perfectly in such idealized models, so markets are complete without them. There is nothing that can be done with options that cannot be done without them.

We analyze a very simple model for trading costs where options turn out to make a large difference. We revisit a Merton optimal dynamic trading problem but now include small proportional transaction costs. Our model includes three assets: a risk free asset we call *cash*, a risky asset we call *stock*, and an *option* with the stock as the underlier. Our four main assumptions are the following:

- A. A trader looks to maximize the expected utility of her wealth at a known later time,  $T > 0$ .

---

\*Received by the editors June 9, 2010; accepted for publication (in revised form) May 11, 2011; published electronically August 9, 2011.

<http://www.siam.org/journals/sifin/2/79805.html>

<sup>†</sup>Courant Institute of Mathematical Sciences, New York University, New York, NY 10012 ([goodman@cims.nyu.edu](mailto:goodman@cims.nyu.edu)).

<sup>‡</sup>Corresponding author. Department of Mathematics and Computer Science, Santa Clara University, Santa Clara, CA 95053 ([dostrov@scu.edu](mailto:dostrov@scu.edu)).

- B. All trades involving the stock or option incur small transaction costs that are proportional to the dollar value of the trade multiplied by a small factor,  $\varepsilon$ .
- C. The stock follows a geometric Brownian motion with known constant expected return and volatility. Cash earns a fixed known return.
- D. There is a single option whose price is determined from its underlier by the Black–Scholes analysis.

If we remove the option from this model, the optimal strategy and its effects have been studied extensively. In particular, [6], [7] give clear arguments for the structure of the optimal strategy, which is to have a *hold region*, named because there is no trading in its interior. At the hold region's boundary, trading occurs so that the portfolio does not escape this region. Rigorous viscosity solution arguments are applied in [18], [12] to the case of a single risky asset, an infinite consumption horizon, and a power law utility function. These papers obtain asymptotic results in the limit of small transaction costs for the optimal hold region strategy. More informal methods, including asymptotic expansions, are applied in a series of papers [20], [3], [4], [2], [16], [10] that expand these asymptotic results to more general utility functions and more than one risky asset. Some of these papers use an infinite consumption horizon; others use consumption only at a final time  $T$ , as we do here. For either consumption model, the form of the asymptotic results, like (1) below, is the same.

Our paper [10] gives a simple heuristic explanation and derivation of these asymptotic results in a very general setting and then shows that this heuristic is equivalent to asymptotic expansion methods. The heuristic explanation discussed at length there and sketched below in section 1.3 shows that the shortfall in expected utility from transaction costs is minimized, to leading order, by balancing a term for the loss from trading costs with a term for the loss from allowing the portfolio to drift from its optimal position. This balancing approach is also used by Rogers in [17] for power law utility functions. The resulting expressions for these two loss terms will form the backbone of some of the analysis of this paper.

With transaction costs but without options, the expected utility,  $f(\varepsilon)$ , has an asymptotic expansion of the form

$$(1) \quad f(\varepsilon) = f_0 - \varepsilon^{2/3}g + \cdots,$$

where the expression for  $g > 0$  can be found quickly from the balancing approach shown in [17], [10]. In this paper, we determine the optimal strategy for using the nondegenerate option and show that this improves the expected utility to

$$(2) \quad f(\varepsilon) = f_0 - \varepsilon^{6/7}h + \cdots,$$

where we determine  $h > 0$  explicitly. The terms  $\varepsilon^{2/3}g$  and  $\varepsilon^{6/7}h$  represent, to leading order, the shortfall in expected utility from transaction costs in the two models. There is an obvious  $O(\varepsilon)$  lower bound on these losses, since one must lose this much to buy or sell the stock or option position even once, such as during liquidation at time  $T$  or when setting up the portfolio initially at time 0. Our  $O(\varepsilon^{6/7})$  costs are much closer to this lower bound than the  $O(\varepsilon^{2/3})$  costs possible without the option. To get a feel for the numbers, if  $g$  were equal to  $h$ ,

then even a reasonably big transaction cost<sup>1</sup> of .3% ( $\varepsilon = .003$ ) would make the trading losses in (2) less than a third of the losses in (1).

Were there an asymptotic expansion for the option price in powers of  $\varepsilon$ , assumption D would be unnecessary because our analysis would use only the leading order term at  $\varepsilon = 0$ , which is the Black–Scholes price. However, there is no expansion for the option price because the incomplete market created by small transaction costs creates deviations to the Black–Scholes price that inherently depend upon the risk preferences of market participants. In this light, assumption D means only that as the transaction costs shrink, we assume the corresponding option price deviations must also shrink along with their effect on the principal shortfall,  $\varepsilon^{6/7}h$ . From a trader’s perspective this corresponds to using the Black–Scholes price as a good first order approximation when transaction costs are small. Of course, the price uncertainty created by transaction costs means that a bid–ask spread in the option market no longer generates an arbitrage opportunity. Assumption B is a simple model for this spread that, as expected, also shrinks as the transaction costs shrink.

We find the analysis here interesting for two main reasons. The first interest is economic: to determine the optimal trading strategy that underlies (2). This strategy identifies an optimal portfolio and trades in reaction to markets to keep the portfolio near a target *magic point*. However, the strategy treats deviations from this magic point in a very anisotropic way. It allows much larger deviations in inexpensive directions along what we call the *Merton line*, than in expensive directions transverse to this line.<sup>2</sup>

The second interest is mathematical: to understand the nature of the asymptotic analysis. We use a three scale asymptotic expansion. The largest scale, which we call *macroscopic*, is order one. The intermediate scale, which we call *mesoscopic*, is order  $\varepsilon^{1/7}$ . The smallest scale, which we call *microscopic*, is order  $\varepsilon^{3/7}$ . The optimal trading strategy allows mesoscopic deviations from the magic point along the Merton line but allows only microscopic deviations away from the Merton line.

We emphasize that this paper presents neither mathematical proofs nor even formal asymptotic expansions. However, in [10], we showed how less formal arguments of the kind offered here are equivalent to the existence of formal asymptotic expansions, at least for the multiple stock optimization problem considered there. Making such formal expansions rigorous, for example using the machinery of viscosity solutions to Hamilton–Jacobi–Bellman equations, seems to be difficult. This was accomplished in [17] and [12] for classes of optimizing single stock portfolios. We are unaware of any rigorous viscosity solution results for the asymptotics of optimizing even a two stock portfolio currently.

<sup>1</sup>As explained in [10] and elsewhere, much of the actual transaction cost of large traders is the bid–ask spread. A \$10 stock with a bid–ask spread of \$.03 would have a .3% proportionate transaction cost in the sense that a round trip—buying and then selling the same share—would cost \$.06 = 2\varepsilon \times (\text{stock price}).

<sup>2</sup>A small analogy can be made between our optimal allocation strategies and option hedging strategies. The optimal allocation strategy of Merton for no transaction costs corresponds to being on the Merton line in this paper. This can be achieved using just stock and cash. Merton’s analysis is closely related to the Delta hedging strategy of Black and Scholes using just stock and cash. In a similar sense, our strategy of using options to further reduce costs near the magic point of the Merton line is analogous to Delta and Gamma hedging strategies. In these hedging strategies, option traders use vanilla options in addition to the underlying stock and cash to hedge exotic options.

**1.1. The model.** We first discuss the model when there are no trades. From assumption C, we have a single underlying stock price process,  $S(t)$ , which is a geometric Brownian motion  $dS = \mu S dt + \sigma S dB$ , with  $\mu$  and  $\sigma$  known and constant. We assume there are no taxes, no dividends, and no carrying costs for short selling. Let  $M(t)$  be the (possibly noninteger, possibly negative) number of stocks held. Then

$$(3) \quad X(t) = M(t)S(t)$$

is the book value of the stock position. If there are no trades in time  $dt$ , then  $dM = 0$  and

$$(4) \quad dX = \mu X dt + \sigma X dB .$$

Let  $Y(t)$  be the value of the cash in the portfolio. From assumption C, if there is no trading, then

$$(5) \quad dY = rY dt ,$$

with  $r$  known and constant.

There also is an option on the stock price. The price of the option at time  $t$  is  $V(S(t), t)$ . From assumption D, we have that  $V$  in our leading order analysis satisfies the Black–Scholes PDE

$$(6) \quad V_t + \frac{\sigma^2 S^2}{2} V_{ss} + rS V_s - rV = 0 .$$

Let  $N(t)$  be the (possibly noninteger, possibly negative) number of options held. Then the book value of the option position is

$$(7) \quad W(t) = N(t)V(S(t), t) .$$

We take the partial derivative,  $W_s$ , to be the sensitivity of  $W$  with respect to  $S$  in the absence of trading, so

$$(8) \quad W_s(t) = N(t)V_s(S(t), t) .$$

Similarly,  $W_{ss} = NV_{ss}$  and  $W_{sss} = NV_{sss}$  later in this paper. If  $dN = 0$  in time interval  $dt$  (no option trading in that time interval), then we have, using (6), that

$$(9) \quad dW = [(\mu - r)SW_s + rW] dt + \sigma W_s S dB .$$

Note that the  $dB$  here and in (4) are the same. We assume that the option has nonzero Delta and Gamma. That is,  $V_s \neq 0$  and  $V_{ss} \neq 0$ . This prevents singular behavior in our analysis. Vanilla European style puts and calls, for example, have nonzero Delta and Gamma and thus make good choices for the option position.

Now we consider the effect of trading on the model. We use the proportional transaction cost model of assumption B. One unit of cash buys  $(1 - \varepsilon a^S)$  worth of stock, where  $a^S$  is a given positive constant; a unit worth of stock can be sold for  $(1 - \varepsilon a^S)$  units of cash; a unit of cash buys  $(1 - \varepsilon a^O)$  units worth of options; and a unit worth of options can be sold for  $(1 - \varepsilon a^O)$

units of cash. Since proportional transaction costs generally model the bid-ask spread, we define the stock price or option price to be close to the midpoint of its respective spread so that the transaction costs for buying or selling a unit of stock are the same, and similarly for the option.<sup>3</sup> The transaction costs for the stock and the option may differ:  $a^S \neq a^O$ . Others have modeled transaction costs as proportionate to the number of shares rather than the dollar value of the transaction. Our model leads to slightly simpler mathematics and, we believe, is equally appropriate as a model of transaction costs faced by large investors.

The assumption that stock and option transaction costs scale with the same parameter  $\varepsilon$  is central to the subsequent analysis. From a modeling point of view, it amounts to assuming that transaction costs for the stock and the option are of a comparable order of magnitude. This is true, for example, in liquid markets for options near to the money on large cap stocks and major indices. From a mathematical point of view, it provides a single expansion parameter, rather than separate parameters for the stock and the option.

Four nondecreasing, nonanticipating processes track the cumulative buying or selling of stock or options:  $I^{BS}(t)$  is the cumulative value of stock bought up to time  $t$ ;  $I^{SS}(t)$  is the cumulative value of stock sold;  $I^{BO}(t)$  is the cumulative dollar value of all options bought; and  $I^{SO}(t)$  is the cumulative value of all options sold. Including these transaction costs in (4), (5), and (9) leads to

$$(10) \quad dX = \mu X dt + \sigma X dB + (1 - \varepsilon a^S) dI^{BS} - dI^{SS},$$

$$(11) \quad dY = rY dt - dI^{BO} + (1 - \varepsilon a^O) dI^{SO} - dI^{BS} + (1 - \varepsilon a^S) dI^{SS},$$

$$(12) \quad dW = [r(W - W_s S) + \mu W_s S] dt + \sigma W_s S dB + (1 - \varepsilon a^O) dI^{BO} - dI^{SO}.$$

Given assumption A, our optimization problem is to choose nonanticipating  $I^{BO}$ ,  $I^{SO}$ ,  $I^{BS}$ , and  $I^{SS}$  to maximize the value function<sup>4</sup>

$$(13) \quad f(x, y, w, t_0, \varepsilon) = \sup_{\mathcal{A}} E^{x, y, w, t_0} [U(Z(T))],$$

where the book value of the portfolio is

$$Z(t) = X(t) + Y(t) + W(t).$$

We assume the utility function,  $U$ , is increasing and strictly concave, specifically,  $U' > 0$  and  $U'' < 0$ . We do not assume a particular form for  $U$  otherwise. The supremum is over all admissible (nonanticipating) trading histories on  $[t_0, T]$  with  $X(t_0) = x$ , etc. Informally, this means that trades at time  $t$  must be functions of the information available at time  $t$ , which is the history  $S(t')$ , for  $t' \leq t$  as well as the trading up to time  $t$ . The rigorous mathematical description of the admissibility constraint is beyond the scope of this paper but is discussed in [7], [18] and the references therein. The principle of dynamic programming and the fact that

<sup>3</sup>This assumption may be relaxed at the expense of making the discussion in section 4 more complicated, for example, making  $u(\eta)$  defined there not a symmetric function of  $\eta$ .

<sup>4</sup>One might argue for using the *liquidation* value  $(1 - \varepsilon a^S)X(T) + Y(T) + (1 - \varepsilon a^O)W(T)$  instead of the book value  $Z(T)$ . The difference between these is  $O(\varepsilon)$ , which is much less than the other losses due to transaction costs, which are  $O(\varepsilon^{6/7})$ .

our model is Markovian imply that optimal trading strategies have the feature that trading decisions at time  $t$  are functions of  $X(t)$ ,  $Y(t)$ ,  $W(t)$ , and  $t$  only. When  $\varepsilon = 0$  (no transaction costs), the value function depends only on the initial total wealth, not the initial allocation. Therefore, we can define the function

$$f_0(z, t) = f(x, y, w, t, 0) ,$$

where  $z = x + y + w$ . With these definitions, our result (2) for the expected utility shortfall can be written as  $f_0(x + y + w, t) - f(x, y, w, t, \varepsilon) = O(\varepsilon^{6/7})$ .

**1.2. Review of portfolios with an option position but no transaction costs: The Merton line for optimal allocation.** We first recall that in the case of no transaction costs and a portfolio with just stock and cash (no options), Merton [15] used the argument of Black and Scholes to show that the optimal stock position, which we call the *Merton position*, is

$$(14) \quad X(t) = m(Z(t), t) , \quad \text{where } m(z, t) = -\frac{(\mu - r)f_{0z}(z, t)}{\sigma^2 f_{0zz}(z, t)} .$$

The Merton function,  $m$ , is always positive since  $f_{0z} > 0$  and  $f_{0zz} < 0$ , which can be established using the fact that  $U' > 0$  and  $U'' < 0$  for all utility functions.

We now review how this result extends to portfolios that also have options on the portfolio's stock. As above, we consider a portfolio with  $M$  shares of stock and  $N$  options. The total portfolio value is

$$(15) \quad Z(t) = Y(t) + M(t)S(t) + N(t)V(S(t), t) .$$

Therefore, from Itô calculus, if there are no trades in time  $dt$  (i.e.,  $dM = 0$ ,  $dN = 0$ ), we have

$$(16) \quad dZ = \left( rZ + M(\mu - r)S + N \left( -rV + V_t + \mu SV_s + \frac{1}{2} \sigma^2 S^2 V_{ss} \right) \right) dt + (M + NV_s) \sigma S dB .$$

The Hamilton–Jacobi–Bellman equation for  $f_0(Z, t)$ , the optimal value function with zero transaction costs, may be written as<sup>5</sup> [15]

$$(17) \quad 0 = \sup_{M, N} E[ df_0 ] .$$

Using the Itô expansion and the formula (16) for  $dZ$  puts (17) in the concrete form

$$(18) \quad 0 = \sup_{M, N} \left\{ \begin{array}{c} f_{0t} + \\ \left[ rZ + M(\mu - r)S + N \left( -rV + V_t + \mu SV_s + \frac{1}{2} \sigma^2 S^2 V_{ss} \right) \right] f_{0z} \\ + \frac{1}{2} (M + NV_s)^2 \sigma^2 S^2 f_{0zz} \end{array} \right\} .$$

---

<sup>5</sup>Throughout this paper we will use abbreviations such as  $E[df_0]$  instead of the more correct form  $E[df_0 | \mathcal{F}_t]$ , where  $\mathcal{F}_t$  is the usual sigma algebra of information available at time  $t$ .

Optimizing the bracketed expression in (18) with respect to  $M$  gives<sup>6</sup>

$$(\mu - r) S f_{0z} + (M + NV_s) \sigma^2 S^2 f_{0zz} = 0 ,$$

which, upon rearrangement, yields

$$(19) \quad M(t)S(t) + N(t)V_s(S(t), t)S(t) = m(Z(t), t) ,$$

where  $m(z, t)$  is the same Merton function given in (14). Alternatively, since  $X = MS$  and  $W_s = NV_s$ , we can express (19) in the form

$$(20) \quad X(t) + W_s(t)S(t) = m(Z(t), t) .$$

In other words, when we add options to our portfolio, we just need to replace  $X$  in (14) with  $X + W_s S$ .

We say that allocations satisfying (19) or, alternatively, (20) lie on the *Merton line* within the plane defined by  $M$  and  $N$  or, alternatively,  $X$  and  $W$ . The expected utility is the optimal value,  $f_0$ , if (19) or (20) is satisfied at every  $t$ . This allows for an arbitrary amount of trading that moves the portfolio along, but never off, the Merton line.

**1.3. Review of small transaction costs without an option position.** Generically, with transaction costs ( $\varepsilon > 0$ ), there is an expected utility shortfall,<sup>7</sup> which is to say that

$$(21) \quad C(x, y, w, t, \varepsilon) = f_0(z, t) - f(x, y, w, t, \varepsilon) > 0 .$$

For small  $\varepsilon$  and  $W = 0$ , it has been shown in increasing generality (see, for example, [20], [17], [18], [3], [10]) that, generically,  $C(x, y, t, \varepsilon) = O(\varepsilon^{2/3})$  as  $\varepsilon \rightarrow 0$ .

This shortfall is due in part to the impossibility (at finite cost) of staying at the Merton position at all times. Without options ( $W = 0$ ), the following *imbalance variable* measures this deviation from the Merton position:

$$(22) \quad \xi(t) = X(t) - m(Z(t), t) .$$

Both without or with options, the optimal strategy<sup>8</sup> [7] is to keep the portfolio within a *hold region*,  $\mathcal{H}$ , about the Merton position (without options) or line (with options) by trading only when the portfolio touches the boundary of  $\mathcal{H}$ . Without options, and for small  $\varepsilon$ ,  $\mathcal{H}$  is approximately described [10] by  $|\xi| < \gamma(Z(t), t)$ .

We now loosely sketch the point of view argued far more concretely in [10] and, for power law utilities, in [17], as this point of view will be central to the perspective we will apply in section 4. In these papers, the optimal  $\gamma$  is found to leading order by minimizing the *shortfall rate*. As explained below, this may be interpreted as the sum of two terms: the *opportunity*

<sup>6</sup>If we optimize with respect to  $N$  instead of  $M$  and then apply (20), we obtain the classic Black–Scholes PDE for  $V$ .

<sup>7</sup>The utility shortfall vanishes only in exceptional cases in which the Merton position does not call for trading, such as all cash or all stock portfolios.

<sup>8</sup>As we said earlier, various authors use slightly different models and optimization criteria. The reasoning described, for example, in [7] for the qualitative form of  $\mathcal{H}$  does not depend on these details.



*loss rate* and the *trading cost rate*. Opportunity loss is the shortfall that comes not from trading but from being off the optimal portfolio. It is a smooth function of  $\xi$  with a minimum value at  $\xi = 0$  (the Merton position), so it is natural that to leading order it is proportional to  $\xi^2$ . Assuming, to leading order, the density of  $\xi$  is uniform<sup>9</sup> in the hold region, we can determine the (leading order of the) opportunity loss rate from the expected value  $E[\xi^2]$ , which is proportional to  $\gamma^2$ .

For the transaction cost, many trades will occur in a short time when  $\varepsilon$  and  $\gamma$  are small, so that we may speak of the trading rate, the amount of trading per unit time. Given that the density is uniform to leading order, this rate is (to leading order) proportional to the density at the boundary, which is  $1/\gamma$ . The trading cost rate, thus, is proportional to  $\varepsilon/\gamma$ .

The total loss rate is the sum of opportunity cost and trading cost rates. Ignoring proportionality constants, this is  $\gamma^2 + \varepsilon/\gamma$ . The optimal  $\gamma$  minimizes this sum, i.e.,  $\gamma_{min} = O(\varepsilon^{1/3})$ . The optimal shortfall rate is on the order of  $\gamma_{min}^2 + \varepsilon/\gamma_{min} = O(\varepsilon^{2/3})$ . It is shown in [10] that this informal analysis is equivalent to earlier arguments, such as in [20], [3], involving asymptotic expansions with a scaling ansatz for the shortfall function,  $C$ .

A more specific quantitative estimate for the trading cost rate is found in [10] from the dynamics for the  $\xi$  variable:

$$(23) \quad d\xi = \eta dB + a dt - dL^+ + dL^-.$$

Here  $L^\pm$  are nondecreasing singular boundary controls satisfying  $dL^+(t) \neq 0$  only if  $\xi(t) = \gamma$  and  $dL^-(t) \neq 0$  only if  $\xi(t) = -\gamma$ . Although  $\eta$  and  $a$  depend on  $z$ ,  $t$ , and  $\xi$ , one can simplify the analysis as follows. The *outer variables*  $Z(t)$  and  $t$  change slowly relative to the time needed for the *inner variable*,  $\xi$ , to reach (approximate) equilibrium with these  $\xi$  dynamics. Further, since  $\xi$  is confined to a small region, it has little effect on  $\eta$  and  $a$ . Thus, to leading order, we can apply the  $\xi$  dynamics in (23) treating  $\eta$  and  $a$  as fixed constants. See, e.g., [8] for a careful discussion of such inner equilibria. Further, as shown in [10], to leading order, the value of  $a$  becomes irrelevant when  $\gamma$  is small. This is because the Brownian motion and the trading on the boundary drown out the effect of the drift as the hold region shrinks. Given this, it turns out (again, see [10], or see the recap in section 3.2) that the trading cost rate is proportional to

$$(24) \quad \frac{\varepsilon \eta^2}{\gamma}.$$

The present work was motivated by the fact that without options,  $\eta = 0$  in (23) when  $m_z = 1$ . (This is easily seen from (22), as, when  $m_z = 1$ , the  $dB$  components of  $dX$  and  $dZ$  are the same, so they cancel.) Most of the earlier papers discussed in the introduction, though not [18], were subject to the generally valid hypothesis  $m_z \neq 1$ , which implies that  $\eta \neq 0$  when options are not present. However, in this paper we will see that using options and the right trading strategies generally enables keeping  $\eta$  close to zero, which substantially reduces the shortfall rate.

---

<sup>9</sup>In [10], it is shown that with a single stock as we have here, the density under the optimal strategy is, to leading order, uniform, but for more than one stock, the density under the optimal strategy is, to leading order, generally far from uniform.



**1.4. Small transaction costs with an option position.** We are now in a position to outline the approach we will take in the remainder of this paper, and we can also explain the scaling ansatz that we will apply in subsequent sections.

By including options, the optimal scalings are altered and the description of  $\mathcal{H}$  becomes more elaborate. For portfolios with options, (20) suggests that the imbalance variable

$$(25) \quad \xi = X + W_s S - m$$

is the natural alteration of the imbalance variable without options given in (22). We note that  $\xi = 0$  corresponds to being on (20), the Merton line of optimal portfolio positions. In section 2, we will find that the dynamics,  $d\xi$ , for this imbalance variable are still of the form given in (23), where the value of  $\eta$  in (23) will be found to be

$$(26) \quad \eta = \sigma S^2 W_{ss} + \sigma(1 - m_z)(m + \xi) .$$

The *magic point* is the position with  $\xi = 0$  (i.e., on the Merton line) that also satisfies  $\eta = 0$ . The point is “magic” because it is the only point on the Merton line where the component of the Brownian motion orthogonal to the Merton line is zero. That is, there is no Brownian diffusive push off the optimal line of portfolio positions at the magic point. Of course, there is diffusion *along* the Merton line, which must be addressed. In section 2, we determine  $d\eta$ , the dynamics for  $\eta$ , in the absence of trading, which will quantify the amount of diffusion we have along the Merton line. We will need to wait until section 3 to quantify the effect that trading has on both  $d\xi$  and  $d\eta$ .

Since  $W_{ss} = NV_{ss}$ , when  $\xi = 0$  we have from (26) that  $\eta = 0$ , that is, we are on the magic point, if we choose

$$(27) \quad N = \frac{(m_z - 1)m}{S^2 V_{ss}} .$$

The denominator does not vanish because we required that  $V_{ss} \neq 0$  for our option. Without an option, if  $\xi = 0$ , then  $\eta = 0$  can be satisfied only in the degenerate situation  $m_z = 1$ .

This observation is the key to understanding why options are so beneficial in our portfolio model, as indicated at the end of the previous subsection. Without options, we are stuck with Brownian motion pushing the portfolio away from its optimal position, except in the singular case where  $m_z = 1$ . With options, however, we now have the freedom to select  $N$  so as to move the portfolio towards the magic point, where the component of Brownian motion that moves the portfolio away from its optimal position vanishes.

The coordinates  $(X, Y, W)$  for the space of portfolios may be transformed, at least locally near the magic point, to the coordinates  $(\xi, \eta, Z)$ . The hold region is given by  $|\xi| < \gamma(\eta, z, s, t)$ , so, since  $\gamma$  shrinks as the transaction costs shrink, the hold region converges to the Merton line. Further, as we will later verify from our optimal trading strategy, the (macroscopic) probability density also converges towards the magic point on this line as the transaction cost shrinks. Therefore, for sufficiently small transaction costs, we stay close enough to the magic point that we can use the coordinates  $(\xi, \eta, Z)$  for our analysis.

These new coordinates allow us to consider more carefully what the scaling of the structure for the optimal strategy should be. Portfolio movement in the  $\xi$  direction, that is, perpendicular to the Merton line, is the least desired, as it generates opportunity loss. Therefore, we

look to strategies whose primary priority is to minimize portfolio movement in  $\xi$ . Movement in the  $\eta$  direction is less destructive since it involves no opportunity loss. However, as  $|\eta|$  increases, destructive Brownian movement in the  $\xi$  direction increases, which suggests that keeping  $\eta$  movement small is the second priority of the optimal strategy.

Given this perspective, we adopt the following scaling ansatz for portfolio movement under the optimal strategy: There are three length scales, starting with the smallest, the *microscopic* scale on which  $\xi$  varies. The new variable,  $\eta$ , varies on an intermediate *mesoscopic* scale. The largest scale is the *macroscopic* scale on which  $Z$ ,  $S$ , and  $t$  vary. This ansatz will be consistent with our results in section 4, where we find that the microscopic scale is  $O(\varepsilon^{3/7})$  and the mesoscopic scale is  $O(\varepsilon^{1/7})$ . The macroscopic scale, of course, is  $O(1)$ .

Our ansatz generates a hierarchy of corresponding time scales for the dynamics that we will apply in section 3. The  $\xi$  dynamics (23) come to (approximate) equilibrium the fastest. On this microscopic level,  $\gamma$  and the variables on which it depends are, to leading order, constant since all of these variables move on larger time scales. (See [8], especially Chapters 3 and 5 and the references therein, for a clear discussion of this and other multiscale analysis properties in a financial context.) As in the case with no options, we will see that the drift coefficient,  $a$  in (23), has no effect on the leading order behavior, and so, as before, we will obtain, to leading order, a uniform distribution for  $\xi$  on the microscopic scale, given values of the mesoscopic and macroscopic variables.

Also in section 3, we will use the results of this microscopic equilibrium to establish that the effective leading order mesoscopic  $\eta$  dynamics, including trading, are

$$(28) \quad d\eta = \kappa dB - \text{sign}(\eta) D \frac{\eta^2}{4\gamma(\eta)} dt .$$

Here the expressions for  $\kappa$  and the positive quantity  $D$ , which are determined in sections 2 and 3.2, respectively, will depend only on macroscopic variables. Therefore, they are constants to leading order at this mesoscopic level. The drift term coefficient,  $D$ , will be shown to depend exclusively on the trading taking place on the boundary. This drift term, unlike the portfolio drift when there is no trading, has an important effect on our analysis because  $\gamma$  in the denominator is small.

For each set of macroscopic variables, there is a mesoscopic steady state probability density for  $\eta$ , which we call  $u(\eta)$ . At the end of section 3, we will apply the Fokker–Planck equation to the mesoscopic dynamics in (28), to quickly establish that, to leading order, this mesoscopic probability density  $u$  satisfies

$$(29) \quad \partial_\eta u + \text{sign}(\eta) \frac{D}{2\kappa^2} \frac{\eta^2}{\gamma(\eta)} u = 0 .$$

In the first part of section 4, we apply the results from our paper [10] to express the leading order expected utility shortfall rate as the sum of the opportunity loss rate and the trading cost rate. Next, we adapt these rate expressions to our mesoscopic context. Specifically, for each value of the mesoscopic and macroscopic variables, the opportunity loss rate is proportional to  $\gamma^2$  and the trading cost rate is proportional to (24). Therefore, at the mesoscopic level, the leading order of the expected utility shortfall rate is the expected value of the sum of these

two positive rates over the (to leading order) steady state probability density for  $\eta$ :

$$(30) \quad A \int_{-\infty}^{\infty} \gamma(\eta)^2 u(\eta) d\eta + \varepsilon B \int_{-\infty}^{\infty} \frac{\eta^2}{\gamma(\eta)} u(\eta) d\eta ,$$

where we will explicitly determine the values of the positive, macroscopic parameters  $A$  and  $B$ .

In the second part of section 4, we look to solve our new form of the optimal trading problem, which is to find a function  $\gamma(\eta)$  that minimizes the total loss rate (30), subject to the constraints (29) and  $\int_{-\infty}^{\infty} u(\eta) d\eta = 1$ . By rescaling this optimization problem to set all coefficients equal to unity, we will determine that the microscopic and mesoscopic length scales are  $O(\varepsilon^{3/7})$  and  $O(\varepsilon^{1/7})$ , respectively. This will also establish that the optimal value of (30) is  $O(\varepsilon^{6/7})$ , as claimed in the introduction.

Finally, in the third part of section 4, we apply calculus of variations to the rescaled optimization problem. The result of our analysis will allow us to numerically determine the optimal  $\gamma(\eta)$  to leading order that defines the optimal hold region,  $\mathcal{H}$ .

**2. Determination of the magic point and its local dynamics.** With positive transaction costs, the continuous trading required to exactly maintain the Merton condition  $X + W_s S = m$  from (20) would be infinitely expensive.<sup>10</sup> Instead, we try to control  $\xi$ , the imbalance in (25). The  $dB$  component of  $d\xi$  is important and simple. We will later show that the  $dt$  component of  $d\xi$  is irrelevant if the hold region is small. This will not be surprising given that the  $dt$  component also became irrelevant when there was no option, as discussed in section 1.3 and shown in [10]. Therefore, we keep careful account of the  $dB$  components in the following calculations for both  $d\xi$  and  $d\eta$ , while we denote the  $dt$  components only by  $adt$ , where  $a$  is allowed to take different values from formula to formula.

Suppose the portfolio is in the interior of  $\mathcal{H}$ , where there is no trading ( $dI^{\text{BO}} = dI^{\text{SO}} = dI^{\text{BS}} = dI^{\text{SS}} = 0$ ). We have that

$$\begin{aligned} d\xi &= d(X + W_s S - m) \\ &= dX + Sd(W_s) + W_s dS - m_z dZ + adt . \end{aligned}$$

Since  $dS = \sigma S dB + adt$ ,  $Sd(W_s) = SW_{ss} dS + adt = \sigma S^2 W_{ss} dB + adt$ , and  $dZ = dX + dY + dW$ , we can use the expressions for  $dX$ ,  $dY$ , and  $dW$  in (4), (5), and (9), along with the definition of  $\xi$  in (25), to obtain

$$\begin{aligned} d\xi &= (1 - m_z) (\sigma X + \sigma S W_s) dB + \sigma S^2 W_{ss} dB + adt \\ (31) \quad &= [\sigma S^2 W_{ss} + \sigma (1 - m_z) (m + \xi)] dB + adt . \end{aligned}$$

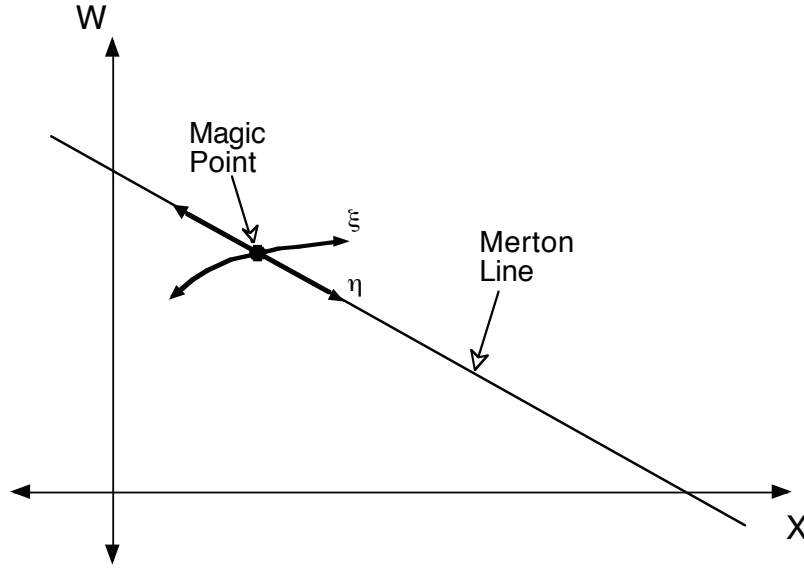
Comparing this to (23) gives the formula for  $\eta$  in (26):

$$\eta = \sigma S^2 W_{ss} + \sigma (1 - m_z) (m + \xi) .$$

The  $dL^\pm$  terms in (23) simply record the fact that there is trading only when  $\xi = \pm\gamma$ . The  $dL^\pm$  terms in (23) vanish when  $dM = 0$  and  $dN = 0$ .

---

<sup>10</sup>Maintaining the Merton condition means choosing  $\gamma(\eta) = 0$ . This corresponds to the second term in (30) for the trading cost rate blowing up.



**Figure 1.** The magic point on the Merton line. The coordinate along the line itself is  $\eta$ . The coordinate transverse to the line is  $\xi$ .

As was previously stated, the magic point (for given macroscopic variable values) is the  $(X, Y, W)$  position that gives  $\xi = 0$  and  $\eta = 0$ . There is a unique choice of  $M$ , the number of stocks, and  $N$ , the number of options, that corresponds to the magic point. First, (27) determines  $N$  and therefore  $W$ . Then (25) determines  $X$  and therefore  $M$ . Finally,  $Y = Z - X - W$ . Figure 1 illustrates the situation. The Merton line corresponds to  $\xi = 0$ . The set  $\eta = 0$  is a line transverse to the Merton line.<sup>11</sup>

As with  $d\xi$ , we compute the  $dB$  coefficient of  $d\eta$  in the interior of  $\mathcal{H}$ , where there is no trading. Differentiating (26) and calculating as before using (4), (5), (9), and (25), we have

$$(32) \quad d\eta = \kappa dB + a dt ,$$

where the value of  $a$  is irrelevant (to leading order), and

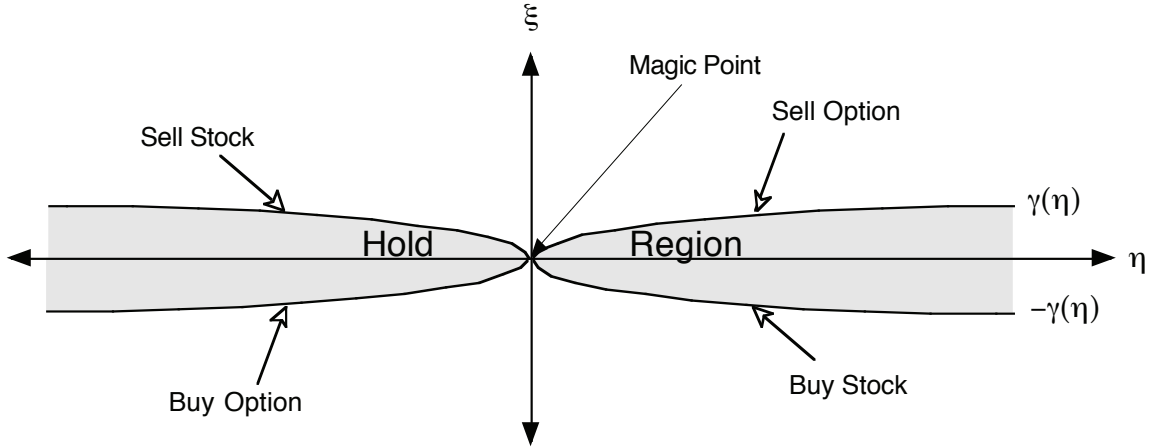
$$\kappa = \sigma \left[ \begin{aligned} &(1 - m_z)\eta + \sigma(m + \xi) \left( -(m + \xi)m_{zz} + (1 - m_z)m_z \right) \\ &+ 2\sigma S^2 W_{ss} + \sigma S^3 W_{sss} \end{aligned} \right] .$$

To leading order,  $\kappa$ , the volatility of the portfolio in the direction parallel to the Merton line, is given by its value at the magic point,  $\xi = 0$  and  $\eta = 0$ . This results in

$$(33) \quad \kappa = \sigma^2 \left( -m^2 m_{zz} + (1 - m_z) m m_z + 2S^2 W_{ss} + S^3 W_{sss} \right) .$$

We will assume that  $\kappa \neq 0$  at the magic point in the remaining sections of this paper. If  $\kappa = 0$  at the magic point, there is no diffusion away from the magic point, which reduces

<sup>11</sup>This follows from the fact that the line  $\xi = 0$  in the  $X, W$  plane has slope  $\frac{dW}{dX} = -\frac{V}{SV_s}$ , whereas the curve  $\eta = 0$  has slope  $\frac{dW}{dX} = -\frac{V}{SV_s} \left( 1 + \frac{SV_{ss}}{V_s(1-m_z)} \right)^{-1}$ , where  $V_{ss}$  is nonzero.



**Figure 2.** A mesoscopic view of the hold region. The hold region extends farther in the  $\eta$  direction than in the  $\xi$  direction. For the trading vectors illustrated, the optimal trading strategy is shown in the figure. In this case, for example, the trading vector for buying stock points up and to the left; it is optimally assigned to the lower right boundary  $\xi = -\gamma(\eta)$ ,  $\eta > 0$ . Therefore, buying stock ( $dI^{BS} > 0$ ) occurs only on the lower right boundary.

transaction costs even further. For nontrivial cases, of course, this idyllic state will not last as the macroscopic variables  $Z(t)$ ,  $S(t)$ , and  $t$  evolve.

There is a parallel between the degenerate case  $m_z = 1$  in portfolios without options and the degenerate case  $\kappa = 0$  in portfolios with an option. Just as we use the option in this paper to reduce costs by exploiting the beneficial singularity when  $m_z = 1$ , we hypothesize that the methods presented here may be extendable to allow further cost reductions using a second option position on the same underlying stock to exploit the beneficial singularity when  $\kappa = 0$ . This two option portfolio may lead to an exponent higher than  $6/7$  for the order of the optimal expected utility shortfall. By the same logic, more than two options may allow the exponent to increase further towards the absolute bound on the exponent of 1. In the interests of simplicity, we do not pursue the advantages of multiple option portfolios further here.

**3. Trading strategy near the magic point.** This section describes the multiscale approximate equilibrium in more detail. We use our ansatz that  $\xi$  varies on a much smaller scale than  $\eta$  so that we can calculate the  $\xi, \eta$  equilibrium in two steps. First, we calculate equilibrium properties of the  $\xi$  dynamics in (23) assuming that  $\eta$  and  $\gamma$  are constants. Then, assuming that  $\xi$  is in approximate equilibrium for each  $\eta$  value, we compute the mesoscopic dynamics, resulting in (28), which we will show immediately implies the differential equation in (29). In section 4 we use this three scale structure to compute the overall utility loss rate. This approximate local equilibrium of the fast variables is discussed at greater length in [8], for example.

The hold region,  $\mathcal{H}$ , is specified analytically by  $|\xi| < \gamma(\eta)$ , where we suppress showing the dependence of  $\gamma$  on macroscopic variables. Determining the optimal  $\mathcal{H}$  is equivalent to finding the optimal function  $\gamma(\eta)$ . The shape of  $\mathcal{H}$  is illustrated in Figure 2. The hold region, to

leading order, is symmetric with respect to  $\xi$  for reasons explained in [10]. The hold region, to leading order, is symmetric with respect to  $\eta$  because of the assumption that the transaction costs for buying and selling are the same. In principle,  $\eta$  is unbounded. However, the leading order expression for the probability density for  $\eta$ , which satisfies (29), will be shown to decay exponentially for  $|\eta| > O(\varepsilon^{1/7})$  in section 4.

**3.1. Analysis of trading on the boundary of the hold region.** Suppose that  $\xi$  is in equilibrium for (23) with a given  $\eta$  and  $\gamma$ , and that  $\gamma$  is small. Then the steady state probability density for  $\xi$  is constant, to leading order, as shown in [10]. This follows just from the fact that trading occurs on the boundary to keep  $\xi$  in the interval  $[-\gamma, \gamma]$ . Otherwise, the specifics of the boundary trading do not matter. On the other hand, the specifics of the boundary trading will influence the effective mesoscopic dynamics of  $\eta$ , as we will show in this section. With a judiciously chosen trading strategy specified in section 3.2 that makes  $D > 0$  in (28), trading will turn out to be responsible for the large restoring force in (28) that pushes  $\eta$  toward the magic point.

The boundary of  $\mathcal{H}$  has four components corresponding to the four quadrants of the  $(\xi, \eta)$  plane. A trading strategy makes a 1-1 association between these four components and the four kinds of trades (e.g., “sell option,” etc.). When the portfolio touches one of the four boundary components, its associated trade is triggered. Figure 2 illustrates one possible assignment.

In the general case, the optimal association of trading type to quadrant depends on the specific nature of the four *trading vectors*. These indicate the direction and magnitude of the changes in  $\xi$  and  $\eta$  corresponding to small amounts of each of the four allowed trades. The trading vectors are  $\mathbf{v}^{\text{BO}}$ ,  $\mathbf{v}^{\text{SO}}$ ,  $\mathbf{v}^{\text{BS}}$ , and  $\mathbf{v}^{\text{SS}}$ . They are explicitly defined by the relation

$$(34) \quad (d\xi, d\eta) = \mathbf{v}^{\text{BO}} dI^{\text{BO}} + \mathbf{v}^{\text{SO}} dI^{\text{SO}} + \mathbf{v}^{\text{BS}} dI^{\text{BS}} + \mathbf{v}^{\text{SS}} dI^{\text{SS}} \\ + \text{terms not involving trading} .$$

We compute these trading vectors to leading order, ignoring terms of order  $\varepsilon$ , whose effect will be taken into account in section 4. Because  $\mathbf{v}^{\text{BO}} = -\mathbf{v}^{\text{SO}}$  and  $\mathbf{v}^{\text{BS}} = -\mathbf{v}^{\text{SS}}$  at the magic (or any other) point, these equalities hold to leading order on the boundary of the small hold region, as illustrated in Figure 2. Therefore, all four vectors are determined by finding  $\mathbf{v}^{\text{BS}}$  and  $\mathbf{v}^{\text{BO}}$ .

We first determine  $\mathbf{v}^{\text{BS}}$ , starting with its  $\xi$  component. We calculate the effect on  $\xi$  from buying a small amount of stock by taking  $dI^{\text{BS}} > 0$  while keeping  $dB = 0$ ,  $dt = 0$ , and  $dI^{\text{SS}} = dI^{\text{BO}} = dI^{\text{SO}} = 0$ . Calculating from (10) and (25) gives simply

$$d\xi = dI^{\text{BS}} + \text{terms of order } \varepsilon .$$

This is because buying stock affects neither  $W$  nor, to leading order,  $Z$ . In the same way, (26) and (10) show that when we buy stock, we get

$$d\eta = \sigma(1 - m_z) dI^{\text{BS}} + \text{terms of order } \varepsilon .$$

Therefore, to leading order,

$$(35) \quad \mathbf{v}^{\text{BS}} = (v_\xi^{\text{BS}}, v_\eta^{\text{BS}}) = (1, \sigma(1 - m_z)) .$$

To find  $\mathbf{v}^{\text{BO}}$ , we compute  $d\xi$  and  $d\eta$  when  $dI^{\text{BO}} > 0$  while keeping  $dB = 0$ ,  $dt = 0$ , and  $dI^{\text{SO}} = dI^{\text{BS}} = dI^{\text{SS}} = 0$ . From (25),  $d\xi = SdW_s$ , where  $dW_s = d(NV_s) = (dN)V_s$ . To determine  $dN$ , we recall that  $W = NV$ , so, for buying options,  $dW = (dN)V$ . But we have from (12) that  $dW = dI^{\text{BO}}$ , so  $dN = dI^{\text{BO}}/V$ . Substituting this expression for  $dN$  yields

$$d\xi = S \frac{V_s}{V} dI^{\text{BO}} + \text{terms of order } \varepsilon.$$

The same reasoning applied to (26) gives

$$d\eta = \left( \frac{\sigma S^2 V_{ss}}{V} + \sigma S(1 - m_z) \frac{V_s}{V} \right) dI^{\text{BO}} + \text{terms of order } \varepsilon.$$

Altogether, we have

$$\mathbf{v}^{\text{BO}} = (v_\xi^{\text{BO}}, v_\eta^{\text{BO}}),$$

where

$$(36) \quad v_\xi^{\text{BO}} = S \frac{V_s}{V}, \quad v_\eta^{\text{BO}} = \frac{\sigma S^2 V_{ss}}{V} + \sigma S(1 - m_z) \frac{V_s}{V}.$$

Note that, by assumption,  $V_s \neq 0$ , so  $v_\xi^{\text{BO}} \neq 0$ . If the option is a put, then  $v_\xi^{\text{BO}} < 0$ ; if it is a call, then  $v_\xi^{\text{BO}} > 0$ . The sign of  $v_\eta^{\text{BO}}$  also depends on the option. All four combinations of signs are possible; that is,  $v_\xi^{\text{BO}}$  and  $v_\eta^{\text{BO}}$  may each be positive or negative.

From (35) and (36) we now have all four trading vectors determined to leading order. The trading vectors are macroscopic to leading order, which is reflected by (35) and (36) exclusively depending on macroscopic quantities.

**3.2. Microscopic and mesoscopic dynamics with trading.** This subsection calculates the mesoscopic  $\eta$  dynamics and the optimal assignment of the four trading types to the four components of  $\partial\mathcal{H}$ . These are related because the correct assignment must enforce  $|\xi| \leq \gamma$  (i.e., keep the portfolio from escaping the closure of the hold region) and, at the same time, push  $\eta$  toward zero in the mesoscopic dynamics (i.e., keep the portfolio near the magic point to reduce diffusion away from the Merton line). The  $\eta$  push comes from the  $\eta$  components of the trading vectors. For easier reference, we label the four components of  $\partial\mathcal{H}$  as *top right* ( $\xi = \gamma(\eta) > 0, \eta > 0$ ), *top left* ( $\xi = \gamma(\eta) > 0, \eta < 0$ ), *bottom right* ( $\xi = -\gamma(\eta) < 0, \eta > 0$ ), and *bottom left* ( $\xi = -\gamma(\eta) < 0, \eta < 0$ ), which conforms with the orientation of the axes shown in Figure 2.

First, we look at the requirement that the association between trading types and components of  $\partial\mathcal{H}$  must enforce that trading on any component of  $\partial\mathcal{H}$  keeps  $|\xi| \leq \gamma(\eta)$ . To ensure this, we must associate trades with  $v_\xi < 0$  at the top and those with  $v_\xi > 0$  at the bottom. For example, (35) implies that  $v_\xi^{\text{BS}} > 0$ , so “buy stock” must be associated with either “bottom right” or “bottom left,” though (35) alone does not say which. Similarly, (36) tells us that if the option is a call, then “buy option” is associated with one of the bottom components of  $\partial\mathcal{H}$ .

Near the origin there will be a small region that vanishes as  $\varepsilon \rightarrow 0$ , where these inequalities for  $v_\xi$  are not sufficient to keep the portfolio in the hold region described by the leading order



expression for  $\gamma(\eta)$  that we will attain. This is because, for this leading order expression,  $\gamma'(\eta)$  will become unbounded as  $\eta \rightarrow 0$ . Therefore, as  $\eta$  shrinks, some macroscopic trading vectors will begin to point out from, instead of into, the  $\mathcal{H}$  defined by our leading order expression for the boundary  $\gamma$ .

To prevent this, we must alter our expression for  $\gamma$  in this small region so that the macroscopic trading vectors continue to point into the hold region. The trading vectors are macroscopic,  $O(1)$ , and  $\gamma'(\eta)$  for our leading order expression is on the microscopic order over the mesoscopic order (shown in section 4 to be  $O(\varepsilon^{3/7})/O(\varepsilon^{1/7}) = O(\varepsilon^{2/7})$ ), which is smaller than the mesoscopic scale of the probability density,  $O(\varepsilon^{1/7})$ . Therefore, on the mesoscopic scale, we have that the region where we must alter  $\gamma$  shrinks.

Our leading order analysis will show that the mesoscopically scaled probability density (given in Figure 4) and its corresponding utility shortfall rate remain bounded as  $\eta \rightarrow 0$ . We make the small assumption that this remains the case when we alter  $\gamma$  in this small region near  $\eta = 0$ . Given this small assumption, we have that this alteration creates a vanishingly small perturbation of our leading order analysis.

Next, we look at the requirement that the association between trading types and components of  $\partial\mathcal{H}$  must create an overall trading effect that pushes  $\eta$  toward  $\eta = 0$ . We noted above that  $\mathbf{v}^{\text{BO}} = -\mathbf{v}^{\text{SO}}$  and  $\mathbf{v}^{\text{BS}} = -\mathbf{v}^{\text{SS}}$ . When we understand the effective  $\eta$  dynamics below, it will be clear that there is no net push toward  $\eta = 0$  if “buy stock” and “sell stock” are both left or both right. Therefore, for a call option we have two possible associations of trade types to boundary components. One, which we will call *possible association 1*, is to have “buy stock” on the bottom right and “sell stock” on the top left, with “buy option” on the bottom left and “sell option” on the top right. The other, which we call *possible association 2*, has “buy stock” on the bottom left and others assigned accordingly. We must understand the mesoscopic effective  $\eta$  dynamics to determine which possible association is correct. The assignments are determined in the same way for a put option.

The mesoscopic dynamics take the form  $d\eta = A(\eta)dt + \Sigma(\eta)dB$ . We find  $A$  and  $\Sigma$  by averaging over the microscopic scale  $\xi$  dynamics in steady state. This means choosing a  $\Delta t$  that is large compared to the  $\xi$  time scale, yet small compared to the  $\eta$  time scale. In terms of  $\Delta\eta = \eta(t + \Delta t) - \eta(t)$ , the  $A$  and  $\Sigma$  are determined by the calculations (see, e.g., [1], [19])

$$E[\Delta\eta] = A(\eta)\Delta t + o(\Delta t)$$

and

$$E[(\Delta\eta)^2] = \text{Var}(\Delta\eta) + o(\Delta t) = \Sigma^2(\eta)\Delta t + o(\Delta t).$$

Only  $A$  turns out to depend on the microscopic  $\xi$  dynamics. In particular, it depends on the boundary forcing necessary to maintain  $|\xi| \leq \gamma$ . Therefore, we calculate  $E[dL^+]$  and  $E[dL^-]$  in the steady state of the microscopic dynamics

$$d\xi = \eta dB + a dt - dL^+ + dL^-$$

from (23). Because of the separation of scales,  $\eta$  may be taken to be constant in this calculation.

At first, let us assume  $a = 0$  in (23). From the steady state, we know that  $0 = d(E[\xi^2])$ . Applying Itô calculus and (23) to this, we have

$$\begin{aligned} 0 &= d(E[\xi^2]) \\ &= E[2\xi d\xi + (d\xi)^2] \\ (37) \quad &= -2\gamma E[dL^+] - 2\gamma E[dL^-] + \eta^2 dt. \end{aligned}$$

Also, from the steady state, we have that  $0 = d(E[\xi])$  and applying (23) to this yields the boundary symmetry  $E[dL^+] = E[dL^-]$ . Therefore, from (37), we have

$$(38) \quad E[dL^+] = E[dL^-] = \frac{\eta^2}{4\gamma} dt.$$

Further, the microscopic dynamics in (23) yield the Fokker–Planck equation for the steady state microscopic probability density,  $p(\xi)$ . This equation is  $p_{\xi\xi} = 0$  subject to the Neumann conditions  $p_{\xi} = 0$  at the boundaries  $\xi = \pm\gamma$ . Solving this equation, we have that  $p$  is constant, and so the steady state distribution of  $\xi$  is uniformly distributed in the interval  $[-\gamma, \gamma]$ .

In the case where  $a \neq 0$  but is bounded, (37) changes to

$$0 = 2E[a\xi] dt - 2\gamma E[dL^+] - 2\gamma E[dL^-] + \eta^2 dt.$$

Because  $|\xi| \leq \gamma$ , this can be written as

$$E[dL^+] + E[dL^-] = \left( \frac{\eta^2}{2\gamma} + O(1) \right) dt,$$

where the  $O(1)$  term contains the effect of  $a$ . However, because  $\gamma$  is assumed to be on a much smaller scale than  $\eta$ , this  $O(1)$  term is lower order than  $\frac{\eta^2}{2\gamma}$  as  $\varepsilon \rightarrow 0$ ,<sup>12</sup> indicating that  $a$  has no effect on the leading order dynamics (see [10] for more). Therefore, even when  $a \neq 0$ , to leading order, (38) holds and the steady state density is uniform.

How the calculation proceeds from here depends on the sign of  $\eta$  and the assignment of trade types to the components of  $\partial\mathcal{H}$ . Assume that  $\eta > 0$  for definiteness and also that we have possible association 1, so “sell option” is on the top right (for a call option) and “buy stock” is on the bottom right. In this case we have, in view of (23) and the  $d\xi$  component of (34),

$$(39) \quad dL^+ = -v_{\xi}^{\text{SO}} dI^{\text{SO}} \quad \text{and} \quad dL^- = v_{\xi}^{\text{BS}} dI^{\text{BS}}.$$

In  $\xi$  equilibrium, (38) combines with this to give

$$(40) \quad E[dI^{\text{SO}}] = \frac{-1}{v_{\xi}^{\text{SO}}} \frac{\eta^2}{4\gamma} dt, \quad E[dI^{\text{BS}}] = \frac{1}{v_{\xi}^{\text{BS}}} \frac{\eta^2}{4\gamma} dt.$$

---

<sup>12</sup>This will be consistent with the scalings determined in section 4. There we will show that  $\eta = O(\varepsilon^{1/7})$  while  $\xi$  and  $\gamma = O(\varepsilon^{3/7})$ , so  $\frac{\eta^2}{2\gamma} = O(\varepsilon^{-1/7})$ , which dominates the  $O(1)$  term generated by  $a$ .

Applying this to the  $d\eta$  component of (34) yields

$$E[d\eta] = \left( \frac{v_{\eta}^{\text{BS}}}{v_{\xi}^{\text{BS}}} - \frac{v_{\eta}^{\text{SO}}}{v_{\xi}^{\text{SO}}} \right) \frac{\eta^2}{4\gamma} dt .$$

Integrating this over a  $\Delta t$  time interval over which  $\eta$  is nearly constant gives the identification (to match the  $D$  notation of (28))

$$(41) \quad A(\eta) = -D_{1+} \frac{\eta^2}{4\gamma} ,$$

with

$$D_{1+} = \left( \frac{v_{\eta}^{\text{SO}}}{v_{\xi}^{\text{SO}}} - \frac{v_{\eta}^{\text{BS}}}{v_{\xi}^{\text{BS}}} \right) ,$$

where the subscripts on  $D$  refer to possible association 1 and positive  $\eta$ .

For  $\eta < 0$ , possible association 1 has “sell stock” on the top and “buy option” on the bottom, which leads to (41) with  $D_{1+}$  replaced by

$$D_{1-} = \left( \frac{v_{\eta}^{\text{SS}}}{v_{\xi}^{\text{SS}}} - \frac{v_{\eta}^{\text{BO}}}{v_{\xi}^{\text{BO}}} \right) .$$

Note that the symmetries  $\mathbf{v}^{\text{BO}} = -\mathbf{v}^{\text{SO}}$  and  $\mathbf{v}^{\text{BS}} = -\mathbf{v}^{\text{SS}}$  imply that  $D_{1+} = -D_{1-}$ .

We now are able to finish determining the assignment of trading types to boundary components. In the case of a call option, possible association 2 corresponds to “sell stock” on the top right and “buy option” on the bottom right, which gives

$$D_{2+} = \left( \frac{v_{\eta}^{\text{SS}}}{v_{\xi}^{\text{SS}}} - \frac{v_{\eta}^{\text{BO}}}{v_{\xi}^{\text{BO}}} \right) = D_{1-} .$$

Putting this together, we have

$$D_{2+} = D_{1-} = -D_{1+} = -D_{2-} .$$

If  $D_{1+} > 0$ , we choose possible association 1, which, from (41), clearly pushes  $\eta$  toward zero when  $\eta \neq 0$ . But if  $D_{1+} < 0$ , then  $D_{2+} = -D_{1+} > 0$ , so we choose possible association 2, which then pushes  $\eta$  toward zero when  $\eta \neq 0$ .

Finally, the possibility  $D_{1+} = 0$  is ruled out as follows. If

$$\frac{v_{\eta}^{\text{BS}}}{v_{\xi}^{\text{BS}}} = \frac{v_{\eta}^{\text{SO}}}{v_{\xi}^{\text{SO}}} ,$$

then  $\mathbf{v}^{\text{BS}}$  and  $\mathbf{v}^{\text{SO}}$  point in the same direction. But the trading vectors in the original  $X, Y$  coordinates were not co-linear, and the smooth invertible change of variables to  $\xi, \eta$  coordinates cannot make linearly independent vectors co-linear. Therefore, we now define  $D$  to equal either  $D_{1+}$  or  $D_{2+}$ , whichever is positive, so  $D > 0$ .

A longer argument, which we omit,<sup>13</sup> shows that, to leading order in  $\varepsilon$ ,  $\kappa = \Sigma$ . Putting all of this together, we have that the effective leading order mesoscopic  $\eta$  dynamics are

$$(42) \quad d\eta = \kappa dB - \text{sign}(\eta) D \frac{\eta^2}{4\gamma(\eta)} dt ,$$

which is (28) from section 1.4. We note that for the unique assignment of trading types to boundary components that both preserves  $|\xi| \leq \gamma(\eta)$  and has  $D > 0$ , it may be the case that either the two stock trading vectors or the two option trading vectors push the portfolio *away* from the magic point, but when this happens, these trading vectors are more than compensated by the other two trading vectors, leading to an overall push towards the magic point.

For each set of macroscopic variables, there is a mesoscopic steady state probability density for  $\eta$ , which we call  $u(\eta)$ . Applying the Fokker–Planck equation to the mesoscopic dynamics just derived in (42), we have that this mesoscopic probability density  $u$  must satisfy

$$\frac{\kappa^2}{2} \partial_\eta^2 u + \text{sign}(\eta) D \partial_\eta \left( \frac{\eta^2}{4\gamma(\eta)} u \right) = 0 ,$$

where  $\partial_\eta$  is the partial derivative operator with respect to  $\eta$ . In view of the boundary condition as  $\eta \rightarrow \pm\infty$ , this integrates to

$$(43) \quad \partial_\eta u + \text{sign}(\eta) \frac{D}{2\kappa^2} \frac{\eta^2}{\gamma(\eta)} u = 0 ,$$

which is (29) from section 1.4.

**4. Optimizing the strategy.** We are ready to examine the problem of minimizing the expected utility shortfall,  $C = f_0 - f$ , in (21). For this, we minimize the positive quantity  $E[-dC]$  in the approximate equilibrium described in the previous section. The dynamic programming principle (see, e.g., [15], [7]) implies that under the optimal strategy,  $E[df] = 0$ . Therefore, (21) implies that  $E[-dC] = E[-df_0]$ , the latter being the Itô differential of a known quantity. We will show that  $E[-df_0] = O(\varepsilon^{6/7})$ . Upon integration up to the final time  $T$ , since  $C(T) = 0$ , this estimate of the shortfall rate implies that  $C = O(\varepsilon^{6/7})$ .

A long but straightforward calculation with the variables above gives an expression for

---

<sup>13</sup>Briefly, let  $\Delta\eta$  be the change in  $\eta$  over  $\Delta t$  that is microscopically large but mesoscopically small. The uncertainty in  $\Delta\eta$  has two sources. One is the direct Brownian forcing term  $\kappa dB$  in (32). The other is the uncertainty in  $\Delta I^{SO}$ ,  $\Delta I^{BS}$ , etc., whose expectations are estimated in (40). This may be estimated using the central limit theorem applied to the one dimensional  $\xi$  process (i.e., the Kubo formula). In the scalings below, one finds  $\text{Var}(\Delta I^{SO}) \sim \varepsilon^{2/7} \Delta t$ , which makes this negligible relative to  $\kappa \Delta t$ .

the shortfall rate:<sup>14</sup>

$$\begin{aligned}
 & -\frac{1}{dt} E [df_0(Z(t), t)] \\
 & = -\frac{\sigma^2}{2} f_{0zz} E [\xi^2] \\
 & \quad + \frac{\varepsilon f_{0z}}{dt} \left\{ a^O (E [dI^{BO}] + E [dI^{SO}]) \right. \\
 & \quad \left. + a^S (E [dI^{BS}] + E [dI^{SS}]) \right\}.
 \end{aligned}
 \tag{44}$$

We omit the details, as they are almost the same as a similar calculation in [10]. Instead, we give some general comments. Both terms on the right are positive since  $f_{0zz} < 0$  and  $f_{0z} > 0$ . Although  $f_0$  depends only on  $Z(t)$  and  $t$ ,  $E[df_0]$  depends on the whole position  $X(t)$ ,  $Y(t)$ , and  $W(t)$  through the implicit  $\mathcal{F}_t$ . Since  $\xi = 0$  minimizes  $E[-df_0]$ , it is natural that  $E[df_0]$  should depend on  $\xi^2$  and  $f_{0zz}$ . The quantity in braces is related to the loss of portfolio value due to trading costs. This translates to loss of utility through  $f_{0z}$ , since  $f_{0z}$  represents the rate of change of the expected utility with respect to portfolio value.

As explained in the introduction, the right-hand side of (44) may be interpreted as the sum of two positive quantities: the opportunity loss rate

$$R_{op} = -\frac{\sigma^2}{2} f_{0zz} E [\xi^2]$$

and the trading cost rate

$$R_{tr} = \frac{\varepsilon f_{0z}}{dt} \left\{ a^O (E [dI^{BO}] + E [dI^{SO}]) + a^S (E [dI^{BS}] + E [dI^{SS}]) \right\}.$$

We evaluate both  $R_{op}$  and  $R_{tr}$  using the microscopic/mesoscopic picture.

To evaluate  $R_{op}$ , we first use the microscopic  $\xi$  equilibrium. Recalling that, to leading order,  $\xi$  is uniformly distributed in  $[-\gamma(\eta), \gamma(\eta)]$ , we have that

$$E [\xi^2 | \eta] = \int_{-\gamma(\eta)}^{\gamma(\eta)} \xi^2 \frac{1}{2\gamma(\eta)} d\xi = \frac{1}{3} \gamma(\eta)^2.$$

Therefore, letting  $u(\eta)$  be the steady state probability density for  $\eta$  under the process (42),

$$R_{op} = A \int_{-\infty}^{\infty} \gamma(\eta)^2 u(\eta) d\eta,$$

where

$$A = -\frac{\sigma^2 f_{0zz}}{6}.$$

---

<sup>14</sup>If  $g(t)$  is a stochastic process with  $E[dg] = Rdt$ , we write the rate  $R$  as  $\frac{1}{dt} E[dg]$ . This notation may be a little nonstandard, but it should be clear.

The trading cost rate,  $R_{tr}$ , is evaluated using similar ideas, this time using (40). The result is

$$(46) \quad R_{tr} = \varepsilon B \int_{-\infty}^{\infty} \frac{\eta^2}{\gamma(\eta)} u(\eta) d\eta ,$$

where

$$B = \frac{f_{0z}}{4} \left[ a^O \left( \frac{1}{|v_{\xi}^{BO}|} + \frac{1}{|v_{\xi}^{SO}|} \right) + a^S \left( \frac{1}{|v_{\xi}^{BS}|} + \frac{1}{|v_{\xi}^{SS}|} \right) \right] .$$

(The absolute values are needed for the two of these four trading components that are negative.) Note that all four trading types contribute to  $B$ , but their contribution is independent of the component of  $\partial\mathcal{H}$  with which they are associated. This completes the derivation of our minimization problem: the optimal trading strategy, to leading order in  $\varepsilon$ , is found by minimizing  $R_{op} + R_{tr}$ , which is the expression

$$A \int_{-\infty}^{\infty} \gamma(\eta)^2 u(\eta) d\eta + \varepsilon B \int_{-\infty}^{\infty} \frac{\eta^2}{\gamma(\eta)} u(\eta) d\eta$$

from (30), subject to the constraint (43) and the normalization constraint  $\int_{-\infty}^{\infty} u(\eta) d\eta = 1$ .

This minimization problem can be rescaled to set all constants equal to 1. The scalings are  $\eta = \alpha\eta'$ ,  $u = \frac{1}{\alpha}u'$ , and  $\gamma = \beta\gamma'$ . The relation between the  $\eta$  and  $u$  scaling preserves  $u'(\eta')$  as a probability density. Substituting into (30) and using the even symmetry in  $\eta$  of both  $u$  and  $\gamma$  yields an equivalent form of the objective function  $R_{op} + R_{tr}$ ,

$$A' \int_0^{\infty} \gamma'^2 u' d\eta' + \varepsilon B' \int_0^{\infty} \frac{\eta'^2}{\gamma'} u' d\eta' ,$$

with

$$A' = 2\beta^2 A , \quad B' = 2\frac{\alpha^2}{\beta} B .$$

Setting the coefficients equal,  $A' = \varepsilon B'$ , gives

$$(47) \quad \beta^3 = \frac{\varepsilon B}{A} \alpha^2 .$$

Scaling the constraint equation (43) gives

$$\frac{1}{\alpha^2} \partial_{\eta'} u' + \frac{D}{2\kappa^2} \frac{\alpha}{\beta} \frac{\eta'^2}{\gamma'} u' = 0 .$$

Setting these coefficients equal gives

$$(48) \quad \beta = \frac{D}{2\kappa^2} \alpha^3 .$$

Together, (47) and (48) lead to

$$(49) \quad \alpha = K_{\alpha} \varepsilon^{1/7} , \quad \text{with } K_{\alpha} = \left( \frac{2^3 \kappa^6 B}{D^3 A} \right)^{1/7} ,$$

and

$$(50) \quad \beta = K_\beta \varepsilon^{3/7}, \quad \text{with } K_\beta = \left( \frac{2\kappa^2}{D} \right)^{2/7} \left( \frac{B}{A} \right)^{3/7}.$$

This establishes our scales for the movement of the portfolio:  $\xi = O(\varepsilon^{3/7})$  and  $\eta = O(\varepsilon^{1/7})$ .

In these scaled variables, the objective function  $R_{op} + R_{tr}$  is (after dropping the primes)

$$K\varepsilon^{6/7} \left( \int_0^\infty \gamma^2(\eta) u(\eta) d\eta + \int_0^\infty \frac{\eta^2}{\gamma(\eta)} u(\eta) d\eta \right),$$

with

$$(51) \quad K = \left( \frac{2^{11} AB^6 \kappa^8}{D^4} \right)^{1/7}.$$

This establishes the  $O(\varepsilon^{6/7})$  cost correction in our primary equation (2).

We next look to determine the optimal strategy, i.e., the optimal scaled trading curve,  $\gamma_{\min}(\eta)$ . Our optimal trading problem has now been reduced to a normalized calculus of variations problem with no free parameters. The problem is to minimize

$$(52) \quad F = \int_0^\infty \left( \gamma^2(\eta) + \frac{\eta^2}{\gamma(\eta)} \right) u(\eta) d\eta$$

subject to the constraints

$$(53) \quad \partial_\eta u + \frac{\eta^2}{\gamma} u = 0$$

and

$$(54) \quad \int_0^\infty u(\eta) d\eta = \frac{1}{2}.$$

There does not seem to be an analytical solution to this minimization problem, but  $F_{\min}$  and  $\gamma_{\min}$  may be computed numerically with the help of the calculus of variations. Our approach follows Appendix 2 of [9] and has much in common with that of Li [13]. We define a Lagrange multiplier function  $\lambda(\eta)$  for the constraint (53) and a constant Lagrange multiplier  $\nu$  for the integral constraint (54). The Lagrangian integral is

$$(55) \quad \mathcal{L} = \int_0^\infty \left[ \left( \gamma^2 + \frac{\eta^2}{\gamma} \right) u - \lambda \left( \partial_\eta u + \frac{\eta^2}{\gamma} u \right) - \nu u \right] d\eta + \frac{\nu}{2}.$$

The first variation of  $\mathcal{L}$  with respect to  $u$  and  $\gamma$  should vanish. We find the variation with respect to  $u$  by the usual integration by parts and the assumption that boundary terms vanish at the limits, in particular,  $\lambda(0) = 0$ .<sup>15</sup> The result of setting this variation to zero is the ODE

$$(56) \quad \partial_\eta \lambda = \nu - \gamma_{\min}^2 - \frac{\eta^2}{\gamma_{\min}} (1 - \lambda(\eta)).$$

---

<sup>15</sup>See [5] for a justification of this point.



It follows from this that  $\nu = 2F_{\min}$  since we can multiply (56) by  $u_{\min}$  (the optimal  $u$ ), integrate, and use (53).

We continue by setting the first variation with respect to  $\gamma$  to zero. After canceling the common factor of  $u_{\min}$  and some other algebra, this gives

$$(57) \quad \lambda = 1 - 2 \frac{\gamma_{\min}^3}{\eta^2}.$$

This implies that  $\lambda(\eta) \leq 1$  since  $\gamma_{\min}$  and  $\eta$  are nonnegative. It also combines with (56) to give an ODE for  $\gamma_{\min}$  in terms of the unknown parameter,  $\nu$ :

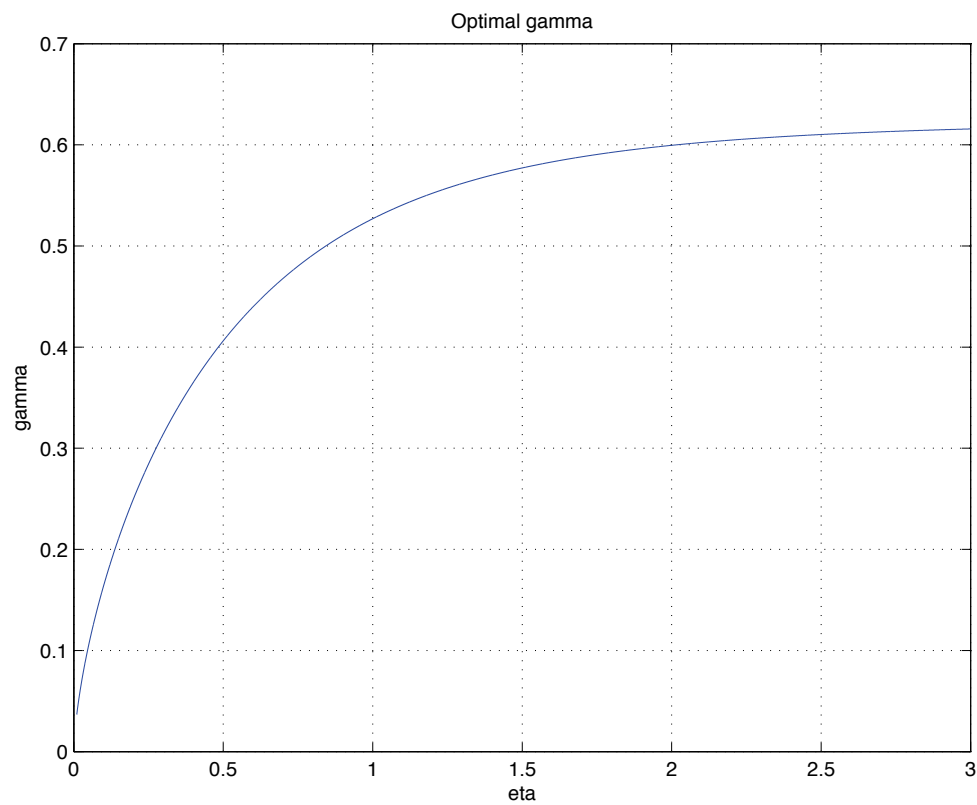
$$(58) \quad \partial_{\eta} \gamma_{\min} = \frac{2}{3} \frac{\gamma_{\min}}{\eta} + \frac{1}{2} \eta^2 - \frac{1}{6} \nu \frac{\eta^2}{\gamma_{\min}^2}.$$

We can characterize  $\gamma(\eta)$  for small  $\eta$  using (58). The first term on the right dominates when  $\eta$  is small. Neglecting the other two terms gives  $\gamma_{\min} \sim C\eta^{\frac{2}{3}}$ . From  $\lambda(0) = 0$  and (57), we see that  $C = 2^{-\frac{1}{3}} \approx 0.794$ . This approximation is self-consistent in that it makes the first term on the right  $O(\eta^{1/3})$  and the last two terms, which were neglected,  $O(\eta^2)$  and  $O(\eta^{2/3})$ .

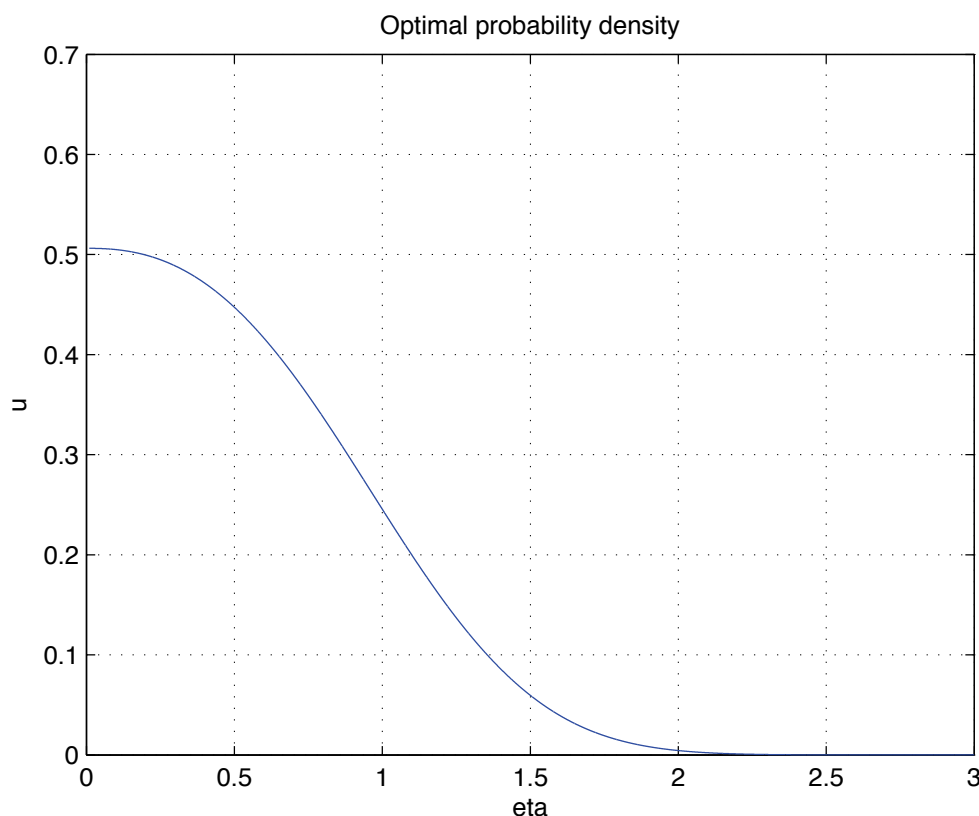
In the numerical solution, we were unable to integrate (58) starting from  $\eta = 0$  because it is unstable. The computed  $\gamma_{\min}$  departs quickly from the true value as  $\eta$  increases. Instead, for a given  $\nu$ , we integrated (58) backwards in  $\eta$  with the condition  $\gamma(\eta_0) = \sqrt{\frac{\nu}{3}}$  for a large  $\eta_0$ . The value  $\sqrt{\frac{\nu}{3}}$  is the asymptotic steady state solution as  $\eta \rightarrow \infty$ .

We find the unknown value of  $\nu$  by trial and error. For a trial  $\nu$  we compute the corresponding  $\gamma_{\min}$  as above, then find  $u_{\min}$  using (53) and (54), and, finally,  $F$  using (52). We found the value of  $\nu$  that minimized  $F$  in this way. Clearly, more efficient methods could be devised.

Given this process, we find that  $\nu \approx 1.170$  minimizes  $F$ , yielding  $F_{\min} \approx 0.5847$ . Note that this fits with  $\nu = 2F_{\min}$ . The corresponding  $\gamma_{\min}$  and probability density,  $u_{\min}$ , for this minimum are given in Figures 3 and 4. Also, as expected,  $\gamma_{\min}(\eta) \sim 2^{-\frac{1}{3}}\eta^{\frac{2}{3}}$  as  $\eta \rightarrow 0$ . This specifies the optimal trading strategy and its associated cost.



**Figure 3.** The graph of  $\gamma_{\min}(\eta)$ , which is the optimal  $\gamma(\eta)$  for the scaled leading order optimization problem (52) under the constraints (53) and (54). After rescaling, this determines the shape of the hold region pictured qualitatively in Figure 2. Only positive  $\eta$  values are shown because  $\gamma_{\min}$  is a symmetric function of  $\eta$ .



**Figure 4.** The steady state probability density function  $u_{\min}(\eta)$  corresponding to the  $\gamma_{\min}(\eta)$  from Figure 3. Only positive  $\eta$  values are shown because  $u_{\min}$  is a symmetric function of  $\eta$ .

## REFERENCES

- [1] L. ARNOLD, *Stochastic Differential Equations: Theory and Applications*, Wiley-Interscience, New York, 1974.
- [2] C. ATKINSON AND P. INGPOCHAI, *The influence of correlation on multi-asset portfolio optimization with transaction costs*, J. Comput. Finance, 10 (2006), pp. 53–96.
- [3] C. ATKINSON AND S. MOKKHAVESA, *Perturbation solution of optimal portfolio theory with transaction costs for any utility function*, IMA J. Manag. Math., 13 (2002), pp. 131–151.
- [4] C. ATKINSON AND S. MOKKHAVESA, *Multi-asset portfolio optimization with transaction cost*, Appl. Math. Finance, 11 (2004), pp. 329–364.
- [5] J. CÉA, *Optimal conception or identification of forms: Rapid calculation of the directional derivative of the cost function*, RAIRO Modél. Math. Anal. Numér., 20 (1986), pp. 371–402.
- [6] M. H. A. DAVIS AND A. NORMAN, *Portfolio selection with transaction costs*, Math. Oper. Res., 15 (1990), pp. 676–713.
- [7] M. H. A. DAVIS, V. G. PANAS, AND T. ZARIPHPOULOU, *European option pricing with transaction costs*, SIAM J. Control Optim., 31 (1993), pp. 470–493.
- [8] J.-P. FOUQUE, G. PAPANICOLAOU, AND K. R. SIRCAR, *Derivatives in Financial Markets with Stochastic Volatility*, Cambridge University Press, Cambridge, UK, 2000.
- [9] J. GOODMAN AND D. N. OSTROV, *Balancing Small Transaction Costs with Loss of Optimal Allocation in Single and Multiple Stock Portfolios*, <http://www.math.nyu.edu/financial-mathematics/content/>

- [02\\_financial/Working\\_Paper\\_Series.html](#) (2007).
- [10] J. GOODMAN AND D. N. OSTROV, *Balancing small transaction costs with loss of optimal allocation in dynamic stock trading strategies*, SIAM J. Appl. Math., 70 (2010), pp. 1977–1998.
  - [11] J. C. HULL, *Options, Futures, and Other Derivatives*, Prentice–Hall, Englewood Cliffs, NJ, 2002.
  - [12] K. JANACEK AND S. E. SCHREVE, *Asymptotic analysis for optimal investment and consumption with transaction costs*, Finance Stoch., 8 (2004), pp. 181–206.
  - [13] L. LI, *Program Trading Strategies in Dynamic Portfolio Management*, Ph.D. thesis, Courant Institute, New York, NY, 2010.
  - [14] J. J. P. MAGILL AND G. M. CONSTANTINIDES, *Portfolio selection with transaction costs*, J. Econom. Theory, 13 (1976), pp. 245–263.
  - [15] R. C. MERTON, *Continuous-Time Finance*, Blackwell, Oxford, UK, 1992.
  - [16] K. MUTHURAMAN AND S. KUMAR, *Multidimensional portfolio optimization with proportional transaction costs*, Math. Finance, 16 (2006), pp. 301–335.
  - [17] L. C. G. ROGERS, *Why is the effect of proportional transaction costs  $O(\delta^{2/3})$ ?*, in Mathematics of Finance, AMS, Providence, RI, 2004, pp. 303–308.
  - [18] S. E. SHREVE AND H. M. SONER, *Optimal investment and consumption with transaction costs*, Ann. Appl. Probab., 4 (1994), pp. 609–692.
  - [19] D. STROOCK AND S. R. S. VARADHAN, *Multidimensional Diffusion Processes*, Springer-Verlag, Berlin, New York, 1979.
  - [20] A. E. WHALLEY AND P. WILMOTT, *An asymptotic analysis of an optimal hedging model for option pricing with transaction costs*, Math. Finance, 7 (1997), pp. 307–324.