A New Approach to Goals-Based Wealth Management

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May 14, 2018

Abstract

We introduce a novel framework for goals-based wealth management (GBWM), where risk is understood as the probability of investors not attaining their goals, not just the standard deviation of investors' portfolios. Our framework is based on a foundation of developments in behavioral economics and finance and is consistent with modern portfolio theory. Using a simple geometric analysis, we determine a specific portfolio that matches each individual investor's stated goals. Our approach requires information from the investor about their goals, elicited in a clear manner that market research shows is superior to common current practices. This new approach can improve the communication between advisors and clients and produce better advice for enabling clients to attain their goals with high probability through the use of efficient portfolios.

1 Introduction

Traditionally, the financial industry, financial advisors, and academics in finance have associated the notion of "risk" with the standard deviation of an investor's portfolio. Investors, on the other hand, typically associate "risk" with the likelihood of not attaining their goals. This distinction is important: for example, decreasing standard deviation risk in an underfunded investor's portfolio increases, as opposed to decreases, the risk of not attaining their goals. We suggest that Goals-Based Wealth Management (GBWM) needs to incorporate viewing risk from the investors' goals-based perspective, as well as the traditional standard deviation perspective.

This has important implications for the portfolios that advisors propose to their clients, as well as for how advisors and their clients communicate. In traditional financial planning, advisors look to understand what an investor's goals are, then they ask questions designed to determine the investor's tolerance for portfolio standard deviation, which leads to advising the investor to adopt a portfolio that has a mean and standard deviation corresponding to the investor's risk appetite, strictly from a standard deviation perspective. In our GBWM approach, we also look to understand what an investor's goals are, but then we seek to elicit what probability the investor would like to maintain in attaining these goals. This is a different conversation, which benefits by using language and ideas that are more natural for investors.

It also leads to different advice for the investor. In this paper, we show how to take the information from this conversation and map it to a specific range of portfolios that will meet the investor's goal-based specifications. In contrast with traditional planning where a static portfolio is generally selected and maintained through rebalancing, our method produces a portfolio that will move about on the efficient frontier, dynamically addressing changes to the market in order to optimize the investor's goals. The trajectory of this evolution will depend on the investor's preferences, which can be pre-selected both in the case when the portfolio is sufficiently funded and when it becomes underfunded if the financial situation worsens sufficiently.

We emphasize that incorporating goals-based wealth management does not mean abandoning risk-based asset allocation. It is an overlay that remains fully consistent with portfolio construction based on modern portfolio theory. GBWM will result in choosing portfolios that reside on the efficient frontier, so they will be optimal, but they accrue the additional benefit of being cognizant of investors' risk in not meeting their goals.

The GBWM approach in this paper has implications for improving the relationship that advisors have with their clients, as well as the outcomes they can obtain for them. The investor is able to benefit not only from individualized advice, but also from being able to explicitly see the effects of choices they understand on the probability of their outcomes. This delivers an experience for the investor that is more intuitive, transparent, and understandable, both in the initial set-up for their investment and for later discussions between the investor and the advisor as market conditions evolve.

The paper proceeds as follows: In Section 2 we describe the antecedent behavioral finance literature and the recent practitioner research on GBWM. Section 3 discusses current practices, introduces nine properties that we believe should be evidenced by GBWM, and also details eight goals-based items of information that investors need to provide our GBWM process. Section 4 presents a simple geometry for GBWM that forms the technical underpinnings for our approach, leading to an investor-specific recommendation for how to invest, both when the investor's portfolio is sufficiently funded or when it becomes underfunded. This new approach has implications for asset allocation, fund selection, and improving the role of financial advisors. Finally, Section 5 offers a concluding discussion.

2 Antecedent Literature

Our approach in this paper is based on an extensive history of academic portfolio theory and behavioral finance research, practitioner-based research, and wealth management experience. We look to combine and extend these different strands into a straightforward, but rigorous, approach.

The 2017 Nobel prize was awarded to Richard Thaler for his work in behavioral economics. In its detailed scientific explanation[1] of the award, the Nobel Prize committee highlighted Thaler's work on the "endowment effect", i.e., the asymmetric valuation of assets by individuals, who value items more when they own them as opposed to when they do not. This is related to loss aversion in Prospect Theory, see Kahneman and Tversky (1979). These psychological constructs are important underpinnings of mental accounting theory (Thaler (1985); Thaler (1999)), where people treat money with different risk-return preferences, depending on what use the money is to be put to, or whether their portfolios are performing poorly or well, Shefrin and Statman (1985). The concepts and ideas in these seminal contributions form the basis for the goals-based wealth management approach developed in this paper.

In its simplest form, goals-based wealth management can be defined as a process that focuses on helping investors realize their goals, both short-term and long-term, through a portfolio management method primarily focused on reaching well-defined financial goals. One of the key underpinnings of GBWM is mental accounting theory. Eliciting investor goals is a key part of the GBWM process, and is facilitated by breaking down overall portfolio goals into sub-portfolio goals using the ideas of mental accounts, where different goals are managed in different accounts, each aggregating into the overall portfolio. Shefrin and Statman (2000) developed behavioral portfolio theory (BPT), and argued that investors behave as if they have multiple mental accounts, a version they titled BPT-MA (in contrast to single account investors, BPT-SA). Each mental account portfolio has varying levels of aspiration, depending on the goals for the mental account. These ideas naturally lead to portfolio optimization where investors are goal-seeking (aspirational), while remaining concerned about downside risk in the light of their goals. Rather than trade off risk versus return, investors trade off goals versus safety, see Roy (1952). This leads to normatively different statements of the portfolio problem than in the mean-variance theory of Markowitz (1952).

These ideas form the bedrock of goals-based portfolio construction, irrespective of whether goals are articulated over single or multiple mental accounts. Interestingly, Das, Markowitz, Scheid, and Statman (2010) showed that, under specific technical assumptions, there is a mathematical linkage between mental-accounting theory (MAT) and mean-variance theory (MVT), arguing that there is a mapping from a goals-based

[1] The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 2017. https://www.nobelprize.org/nobel_prizes/economic-sciences/laureates/2017/advanced-economicsciences2017.pdf.

portfolio to a portfolio on the mean-variance efficient frontier. This mathematical reconcilement showed that GBWM is supported by MVT, which also forms part of the basis for the model described in this paper.

There is a growing practitioner literature on goals-based wealth management. Nevins (2004) extended the mental accounting approach. He contended that traditional investment planning fails to recognize investors' behavioral preferences and biases, resulting in suboptimal portfolio performance. He argued that traditional risk measures do not fully capture market behavior and are of limited relevance to investors. His paper advocated that, in addition to investors' behavioral attributes, such as mental accounting and loss aversion, a goals-oriented approach helps investors ameliorate their biases, such as overconfidence, hindsight bias, overreaction, belief perseverance, and regret avoidance, all identified by researchers in behavioral finance.[2] Complementing this work, Zwecher (2010) discusses how risk management can be done more actively and efficiently by demonstrating how a retirement portfolio that provides income, generates growth, and protects assets from disasters, can be created by adopting a bucketing (mental accounting) approach.

Brunel (2015) discussed the equal importance of two goals for an investor: being able to avoid nightmares while realizing dreams. Brunel's work focussed on demonstrating how goals-based wealth management can be achieved across multiple time horizons for multiple life goals. He also suggested how to map the language investors use in describing the importance of dreams or the severity of nightmares into acceptable probabilities that the investor will realize such dreams or avoid such nightmares.

Based on the research above, our approach in this paper follows from the idea that investors are better able to articulate and discuss their goals, including safety criteria, than they are able to specify a mean-variance trade-off and that working with goals leads to portfolios that are better designed to meet investor aspirations. Practitioners have recognized the need for a goals-based approach, and this paper offers a framework and implementation model for a single goal, showing how this goals-based portfolio approach is managed over time in a manner that also fully supports traditional asset-allocation approaches.

In the next section, we discuss how our approach differs from typical current approaches used by many advisors.

[2] See the following excellent books that cover many of these biases: (i) Shefrin (2007); (ii) Statman (2010); (iii) Shefrin (2016). These ideas are embodied in a concept paper by Chabbra (2005).

3 Properties of the Goals-Based Wealth Management (GBWM) Approach for Financial Advising

As we see from the previous section, GBWM strategies can offer a rich, normative structure with many attractive properties. These properties have practical value in supporting a new approach to financial advising:

Property 1: *Clarity.* The financial goals of the investor should be clear.

Property 2: *Customization.* The advice given to the investor should be individualized to cater to their specific goals.

Property 3: *Risk Specificity.* The probability of the investor attaining (or not attaining) their goals should always be clear.

Property 4: *Risk Compliance.* The advice given to the investor should take into account these probabilities.

Current strategies that are intended to be goal-based usually satisfy the first property, but the three other properties are often not met. Consider, for example, the following four common approaches currently taken by advisors looking to be goal-based:

1. The financial advisor helps investors to elaborate their broad investment goals and sub-goals. They consider the future value of the money for these goals adjusted for inflation. Money is then allocated to these goals. This approach meets the first two properties, but not the third or fourth.[3]

2. The financial advisor helps investors create risk-based portfolios depending on how far ahead in the future their goals are, and accordingly adjusts their equity exposure. For example: 70% equity exposure for a goal 20 years away, 60% for 15 years, and so on. This approach meets the first property and may be executed in a way that meets the third property, but it does not meet Property 4, nor, since the advice in a given time frame is the same regardless of the size of the goal wealth, does it meet Property 2.

3. After the advisor determines the investor's financial goals, each goal is mapped to a specific investment portfolio (e.g., a "must have" goal is mapped to the least volatile investment portfolio, a long term investment is mapped to a risk-adjusting glide path portfolio, etc.). Again, this approach meets the first property and may be executed in a way that meets the third property, but it does

[3]Advisors are mostly focused on long-term goals such as "not running out of money" or "keeping one's home." To this end, retirement is clearly the most encompassing goal, followed by retirement income planning. And, as a result, advisors say their solutions are very focused on tax strategies (taxable versus tax-free versus tax-deferred assets) and time horizons.

not meet Property 4, nor, since the size of the goal wealth is not taken into account, does it meet Property 2.

4. In addition, a financial advisor may also offer expense management solutions, enabling them to advise investors on how to cut back expenses or streamline monthly budgets, incorporating present and future cash flows, so the investor can achieve certain savings goals. While these are an important part of financial planning, the objective of expense management is different from the objective of goals-based wealth management, so it is unsurprising that this approach meets only the first two properties, but not the third or fourth.

The first and the last of these example approaches are disconnected from the equities market. The second and third approaches look at risk strictly as the standard deviation of the portfolio, not the probability of failing to achieve the investor's goals. Because risk is viewed by many advisors strictly as portfolio standard deviation and because investment success is often defined by advisors as individual investments' outperformance of their benchmarks, instead of a portfolio meeting its goal or goals, advisors often view the success (or lack of success) of a portfolio in different ways from investors. The crucial components in GBWM of considering the probability that the investor's goals will actually be met, encompassing all four properties, is by and large, missing in current practice. The GBWM process also has to address "how" the goals will best be achieved. Investors need to know how close or far they are to achieving their goals, and whether they are protected in the event of adverse market events.

Market surveys on outcome-oriented investment research allow us to better understand and quantify this issue. For example, consider Figure 1, from a recent market survey of investors and advisors,[4] which shows that 72% of clients view success based on the overall performance of their portfolio, not the performance of individual investments, whereas only 51% of advisors believe that their clients view success primarily based on overall portfolio performance, as opposed to individual investment performance.

Further, by focusing more on individual investments, instead of the overall goals of the portfolio, the advisor will gravitate towards language and ideas that are less clear to the client, as Figure 2, which comes from the same recent market survey, demonstrates. Talking with a client in goals-based probability language such as "Based on your current strategy, there is a 90% chance that you will achieve your investment goal." is very clear to 49% of clients and either very clear or quite clear to 92% of clients, whereas individual investment oriented language like "Your U.S. equity investment has been outperforming its benchmark index." is only very clear to 29% of clients and either very clear or quite clear to 71% of clients.

This is a key observation: clients are more comfortable with discussing and thinking about their goals and the probability of attaining them, and advisors who adopt

[4]Franklin Templeton partnered with Hall and Partners to conduct this survey of 300 advisors and 503 investors in May 2017.

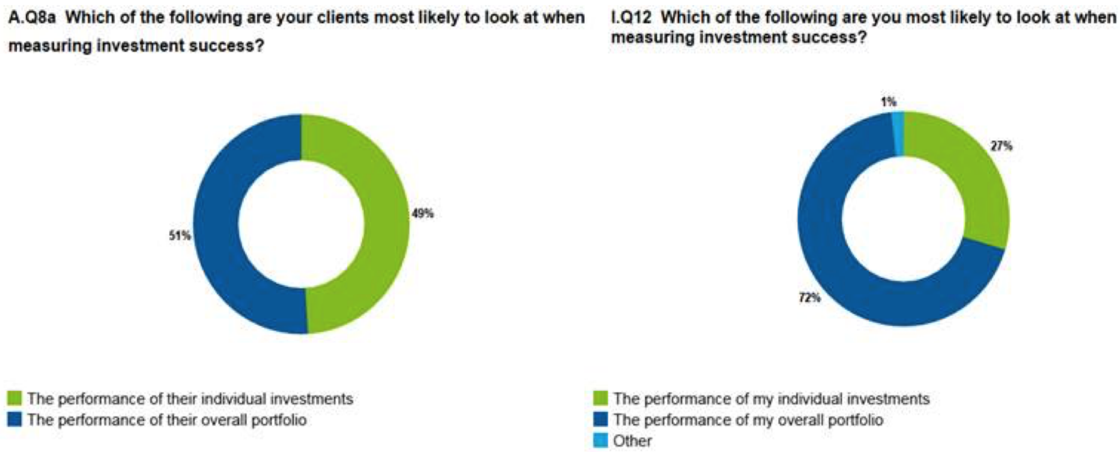

Figure 1: Responses of advisors and investors to a survey on the investors' portfolio performance.

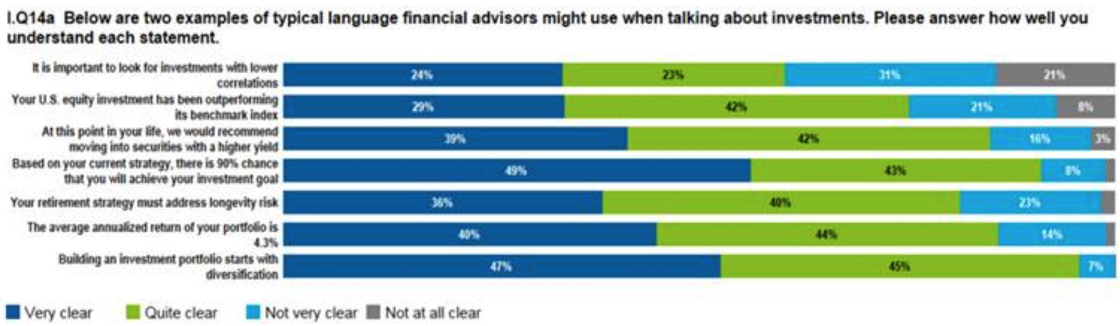

Figure 2: Responses of investors to a survey on their goals.

this goal-based language will be better able to communicate with their clients and also be in a better position to consider how to address their clients' needs. Therefore, in addition to the previous four stated properties, GBWM should have the following property:

Property 5: *Client-centered Communication.* Clients should be asked for information about their goals in terms that are clear to them, such as investment time frames, desired dollar amounts at the end of these time frames, and desired probabilities of attaining these dollar amounts. Both the initial and future interactions between advisors and clients benefit by a method enabling discussions that can, if desired, exclusively use clear, relatable, goals-based language.

Discussing goals with investors involves understanding their target wealth and the desired probability of attaining that wealth, which speaks to the aspiration-related goals of the investor (see Lopes (1987)). But it also involves understanding their loss threshold wealth and the desired probability of not ending up below this loss threshold wealth, which speaks to the fear-related goals of the investor (Shefrin and Statman (2000)). The advisor plays a crucial role in working with the investor to determine these probabilities. One example for how an advisor can help translate the investor's qualitative desires into specific probability values is by using the goals-based framework of Brunel (2015). This framework, represented in Table 1, associates the investor's needs, wants, wishes, or dreams of achieving their target goal to associated probabilities, as well as associating the investor's nightmares, fears, worries, or concerns about ending below their loss threshold to probabilities.

Table 1: Brunel's Goal-Probability table can be used to classify investor goals into different probability values, which can then be used in our GBWM framework.

Realize	Avoid	Success Probability
Dreams	Concerns	50% 55% 60%
Wishes	Worries	65% 70% 75%
Wants	Fears	80% 85%
Needs	Nightmares	90% 95%

Within the GBWM approach that we develop here, Prospect Theory (Kahneman and Tversky (1979)) is modified. Prospect Theory specifies a single "reference point" of return, and investors behave as if they have different risk preferences when they are above the reference point versus when they are below it. For example in the case of Shefrin and Statman (1985), the disposition effect implies that people hold onto losers too long and sell winners too early. In our implementation of GBWM, instead of one reference point, we have two, (i) a target wealth on the spectrum of upside outcomes, and (ii) a threshold wealth for losses on the downside. The framework in this paper will present an implementation approach for achieving investor goals, taking into account the probabilities for target goals to be met and the risk of falling below acceptable loss thresholds.

Qualitative research including discussions with dozens of financial advisors,[5] many of whom use one or more of the four common approaches described above, uncover

[5] Franklin Templeton partnered with aQity Research & Insights, Inc. to conduct qualitative research from September–December 2017. Independent qualitative research was also conducted over the same period.

a number of observations from the advisors concerning current goal-based strategies, such as the following:

1. Although clients go through an elaborate financial planning process by answering detailed goals-based questionnaires, portfolio construction generally reverts to one of a finite number of standard asset allocation models, and the allocation model chosen is determined primarily by the time horizon for achieving the client's goal.

2. In evaluating a portfolio's performance, standard financial industry performance indicators are used, such as excess returns, alpha, tracking error, R-squared values, and information ratios, which are only relevant for comparing individual investments to their benchmarks.

3. Once the investment strategy is developed, one investment may be replaced by a similar investment within a sector, with the assumption that the new investment would maintain the same interrelationships with the other investments in the portfolio.

4. Underfunded portfolios are often ignored, resonating with an often heard criticism in the wealth and asset management industry that financial advisors are "doctors who treat only healthy patients."

5. Critical goals are often over-funded, consuming too much of the investor's principal.

6. Periodically, portfolios are adjusted to bring them back to a target asset allocation, regardless of market conditions.

7. While advisors understand the concept and intuitions of goals-based wealth management, they do not have a framework to analyze its implications for portfolio construction.

While a GBWM method with the above 5 properties addresses some of these observations, it does not address all of them. Therefore, we incorporate the following GBWM properties:

Property 6: *Goal/State Specificity.* Advice discussed with an investor should be based on information regarding the investor's goals and the overall state of their portfolio, not, in general, the portfolio's individual investment components.

Property 7: *Portfolio Efficiency.* Investors should always be advised to invest in a portfolio that is on the Efficient Frontier.

Property 8: *State Dependency.* The advice given to investors (that is, the specific location on the Efficient Frontier) should be affected by market changes and easy to understand investor preferences, both in bull and bear markets.

Property 9: *Rules for Rebalancing.* The investor's portfolio should be able to be updated automatically at regular intervals (e.g., annually, so as to avoid short-term capital gains) and manually, whenever the investor wishes.

GBWM Properties 4, 5, and 6 help address the first observation of the advisors in the focus group. GBWM Property 6 also addresses the second observation, while Property 7 addresses problems that can arise from the third observation, and Properties 8 and 9 help address the fourth, fifth, and sixth observations.

In the next section we detail a portfolio process that satisfies all nine of these GBWM properties and also addresses the seventh noted observation of advisors, by providing specific portfolio advice for clients from their answers to simple, goals-based information, following GBWM Property 5. Specifically, we first ask clients to specify the following basic information:

1. Their time frame (Investment Horizon).
2. The size of their initial investment (Initial Wealth).
3. Their goal wealth (Target Wealth).
4. The probability they would like to maintain of reaching their goal wealth (Target Probability).
5. The wealth they would not want to end below (Loss Threshold Wealth).
6. The probability they would like to maintain of ending above the loss threshold (Loss Threshold Probability).

Following GBWM Property 7, we will initially require that there is a range of portfolios (or at least one portfolio) on the Efficient Frontier that meets all of these investor specifications. If the portfolio does well, we will refer to the portfolio as being in a "good" state for the investor. Should the market not do so well, including evolving to cases where no investment on the frontier meets all of the investor's specifications, we will refer to the investor's state as "bad." The specifics of these definitions will be given in the next section.

Investors may have different priorities in good states and in bad states, so we will ask them for two final pieces of information to understand these priorities:

7. Their investment preferences in good states, and
8. Their investment preferences in bad states.

For good states, this means having the investor specify if they would like to (i) pursue a strategy that optimizes the probability of meeting their target wealth, (ii) pursue a strategy that, subject to conforming to their target and loss threshold specifications, optimizes their average portfolio returns, or (iii) pursue a strategy somewhere between these two strategies.

For bad states, the choices are different, with the investor specifying if they would like to (i) pursue a strategy that optimizes the probability of meeting their target wealth, (ii) pursue a strategy that optimizes the probability of maintaining their loss threshold wealth, or (iii) pursue a strategy somewhere between these two strategies.

We show how to determine a portfolio using our GBWM approach in the next section by just using these eight pieces of information.

4 Implementation Geometry

4.1 Overview

Because our view of risk is connected to the probability of not achieving goals, instead of just the volatility, σ, of an investor's portfolio, the approach we take in this section will differ significantly from current common approaches to goals-based investing. Our approach will clearly connect how the probability of attaining an investor's goals corresponds to a specific interval of risk-return combinations on the Efficient Frontier. As we progress upwards through this interval on the Efficient Frontier, the mean and volatility of the portfolio increases. The probability of the investor attaining their goals also changes, but not in a way that has been previously well understood. We will show, for example, that the traditional notion of gravitating towards the least volatile portfolio in this interval generally does not correspond to the safest choice for the investor. Again, this is because the notion of "safest" is defined by the probability of attaining their goal, whereas traditionally it has been defined without this goals-based perspective.

In Subsection 4.2, we explore the three building blocks needed to determine this interval on the Efficient Frontier that meets the investor's specifications. Subsection 4.3 shows how to put these building blocks together to determine this interval and how, based on the client's investment preferences in good and bad states, to determine the specific point on the frontier that corresponds to the advice given to the client for how to invest. Finally, in Subsection 4.4 we explain some insights and observations about our approach, and in Subsection 4.5 we discuss some aspects of how to apply our approach in practice.

4.2 Three Building Blocks

There are three primary building blocks in our approach: (1) The Goal Region and its associated Goal Probability Level Curves, (2) The Loss Threshold, and (3) The Efficient Frontier of available investments. Geometrically, each of these will correspond to regions or curves in the plane where the risk, σ (i.e., portfolio volatility[6])

[6]We will use the term "risk" to mean portfolio volatility in this section, because this has been the historical definition, although our approach, of course, continues to focus on thinking about risk as the chance that an investor will not meet their goals, as well as the portfolio volatility.

is on the horizontal axis and the return, μ (i.e., portfolio expected return) is on the vertical axis.

4.2.1 Goal Probability Level Curves (GPLC)

Recalling the information we have collected, an investor has specified an Investment Tenure, t, by which time they want their specified Initial Wealth, $W(0)$, to grow at least to their specified Target Wealth, $W(t)$, with a specified Target Probability. Under the model of geometric Brownian motion, we know that future wealth, $\tilde{W}(t)$, is given by

$$\tilde{W}(t) = W(0)e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\sqrt{t}Z}, \tag{1}$$

where Z is a standard normal random variable. Rearranging equation (1) and replacing $\tilde{W}(t)$ with the Target Wealth, $W(t)$, yields the key relationship

$$\mu = \frac{1}{2}\sigma^2 + \frac{z_0}{\sqrt{t}}\sigma + \frac{1}{t}\ln\left(\frac{W(t)}{W(0)}\right), \tag{2}$$

where z_0 is defined so that the Target Probability equals $\Phi(z_0)$, with $\Phi(z)$ being the cumulative distribution function (CDF) for a standard normal random variable. Note that equation (2) defines an upward curving (i.e., convex) parabolic relationship between the expected return μ and the volatility σ.

Any portfolio lying on or above this parabola in the (σ, μ) (i.e., risk-return) plane will satisfy the investor's stated goals, therefore, the region on or above this parabola is called the **Goal Region**. The parabola itself is the **Boundary of the Goal Region**. If we replace z_0 with a generic value z in the above equation, we call the resulting parabola a **Goal Probability Level Curve** (GPLC) corresponding to the probability that a normal random variable will take a value less than z. For example, an 80% Goal Probability Level Curve will be the parabola corresponding to the set of all (σ, μ) pairs where there is exactly an 80% chance that the investor will meet or exceed their goal (see Figure 3).

We note that equation (2) above only depends on $W(0)$ and $W(t)$ through their ratio. The goal region in Figure 3 corresponds to an increase over an investment tenure of 10 years from an Initial Wealth, $W(0)$, of \$400,000 to a Target Wealth, $W(10)$, of \$500,000. This corresponds to a continuous rate of return of 2.23% per year. We retain the same goal region in Figure 3 for any other $W(0)$ and $W(10)$ pair that corresponds to this same 2.23% rate of return, such as, say, $W(0) = \$200,000$ and $W(10) = \$250,000$, since the ratio, $W(10)/W(0)$, stays the same.

We also note that if we increase z, which increases the percentage for the GPLC, the parabolic curve moves upwards in the (σ, μ) plane everywhere except where the curve meets the μ axis. On the μ axis, $\sigma = 0$, so from equation (2) we have that

$$\mu = \frac{1}{t}\ln\left(\frac{W(t)}{W(0)}\right), \tag{3}$$

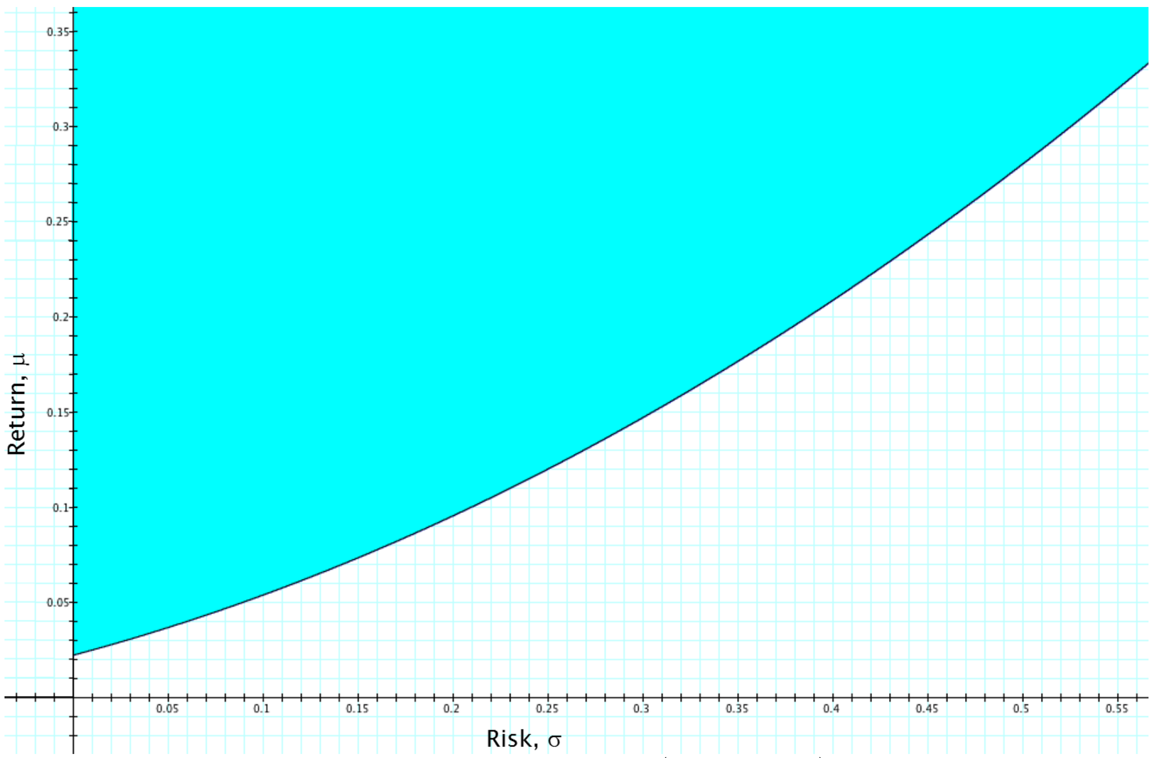

Figure 3: A Goal Region Example. The goal region (in light blue) in this figure corresponds to a base case where an investor has an Investment Tenure of 10 years, an Initial Wealth of \$400,000, a Target Wealth of \$500,000, and a Target Probability of 80%. The boundary of the goal region is the 80% Goal Probability Level Curve (GPLC), which is defined as the curve for which any (σ, μ) pair above the curve corresponds to a portfolio with an 80% or higher chance of the investor attaining their Target Wealth at the end of their investment tenure.

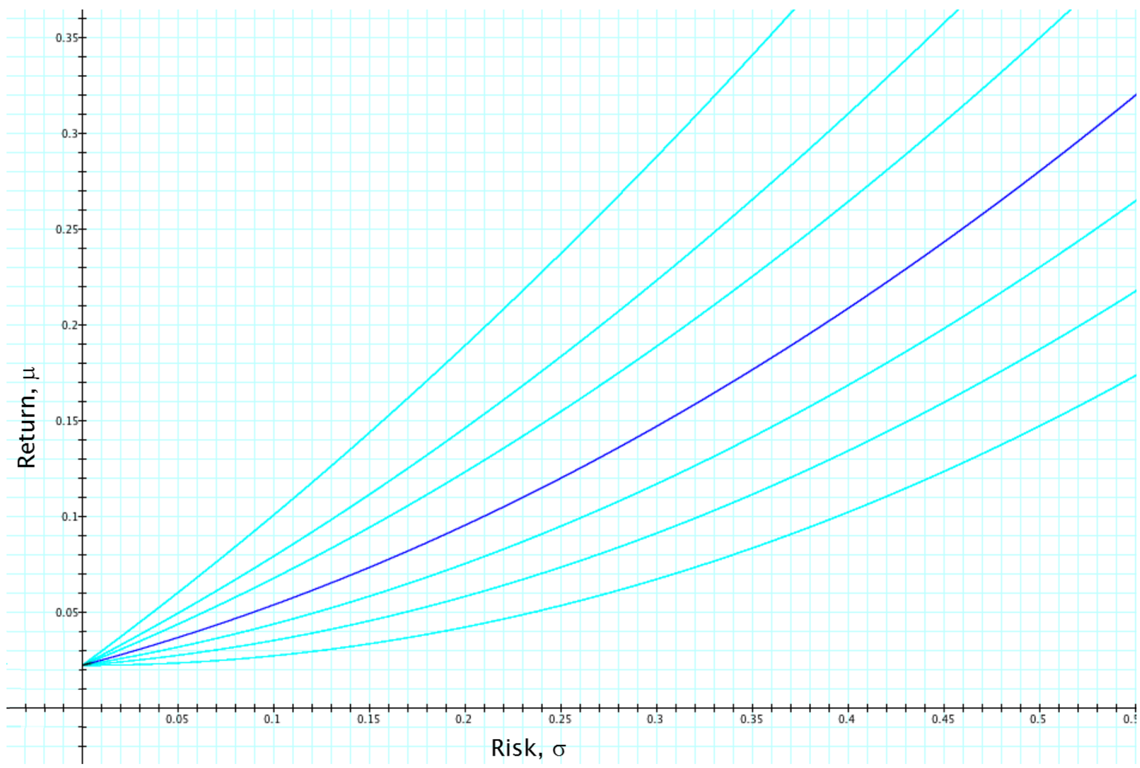

Figure 4: The Dependence of Goal Probability Level Curves on Probability Levels. The top Goal Probability Level Curve (GPLC) in this figure is the 99% GPLC. Below that are the 95%, 90%, 80% (in dark blue, denoting the base case), 70%, 60%, and 50% GPLCs. The Investment Tenure, Initial Wealth, and Target Wealth parameters for these GPLCs are the same as in the base case given in Figure 3.

regardless of the value of z. In other words the Initial Wealth, the Goal Wealth, and the Investment Tenure determine the location at which all the GPLCs meet the μ axis, while increasing the percentage for the GPLC moves the curve upwards, except at this fixed point on the μ axis. By varying z, we generate a family of GPLCs, as we show in Figure 4.

We can look at the effect on the GPLC of varying other parameters from their base case values, such as the Target Wealth (See Figure 5) or the Investment Tenure (See Figure 6).

4.2.2 Loss Threshold Curves (LTC)

In addition to their Target Wealth and Target Probability, investors also specify a Loss Threshold Wealth along with a Loss Threshold Probability. Recall that the Loss Threshold speaks to the investor's fears, while the Target speaks to the investor's dreams. The intent of the investor is to stay at least at or above their Loss Threshold Wealth at the end of their Investment Tenure with a probability equal to or higher

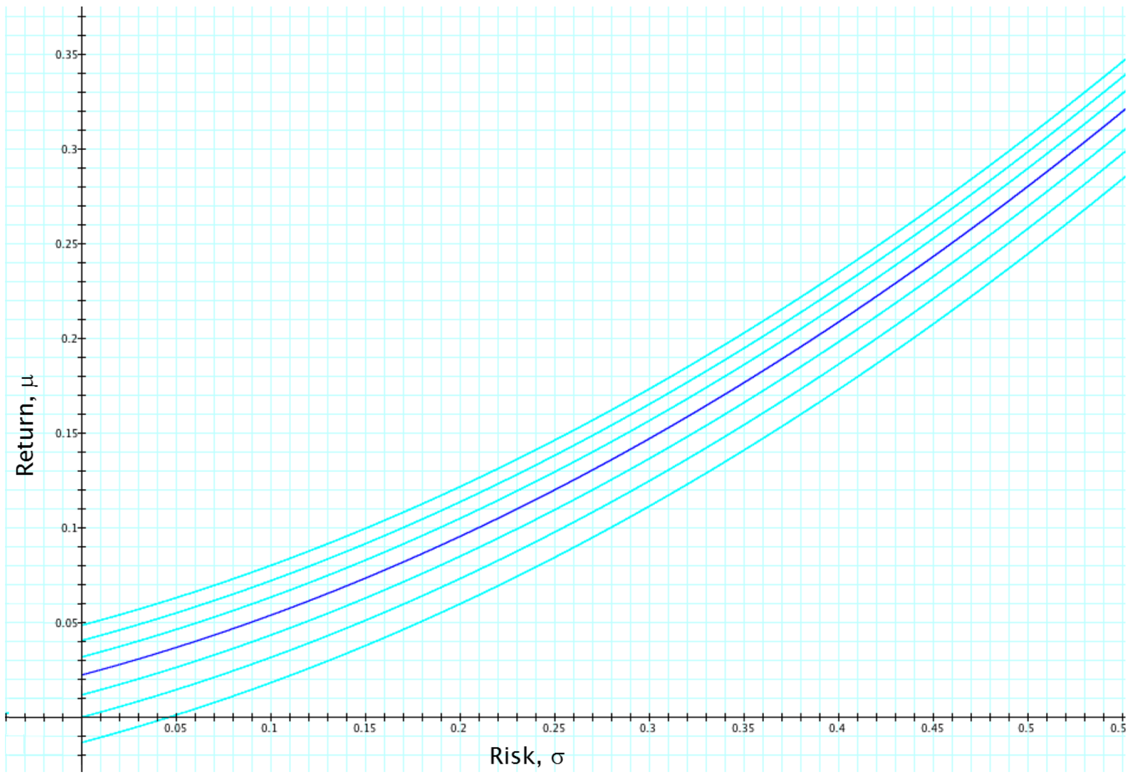

Figure 5: The Dependence of Goal Probability Level Curves on Target Wealth. The top GPLC corresponds to a Target Wealth of \$650,000, instead of the base case value of \$500,000. Below that are the GPLCs as the Target Wealth is reduced to \$600,000, \$550,000, \$500,000 (in dark blue, again denoting the base case), \$450,000, \$400,000, and \$350,000. We note that these GPLCs are all parallel curves. The GPLC curve for a Target Wealth of \$400,000 goes through the origin, since the Initial Wealth is also \$400,000. As in Figure 4, all parameters other than the Target Wealth have their base case values, as given in Figure 3.

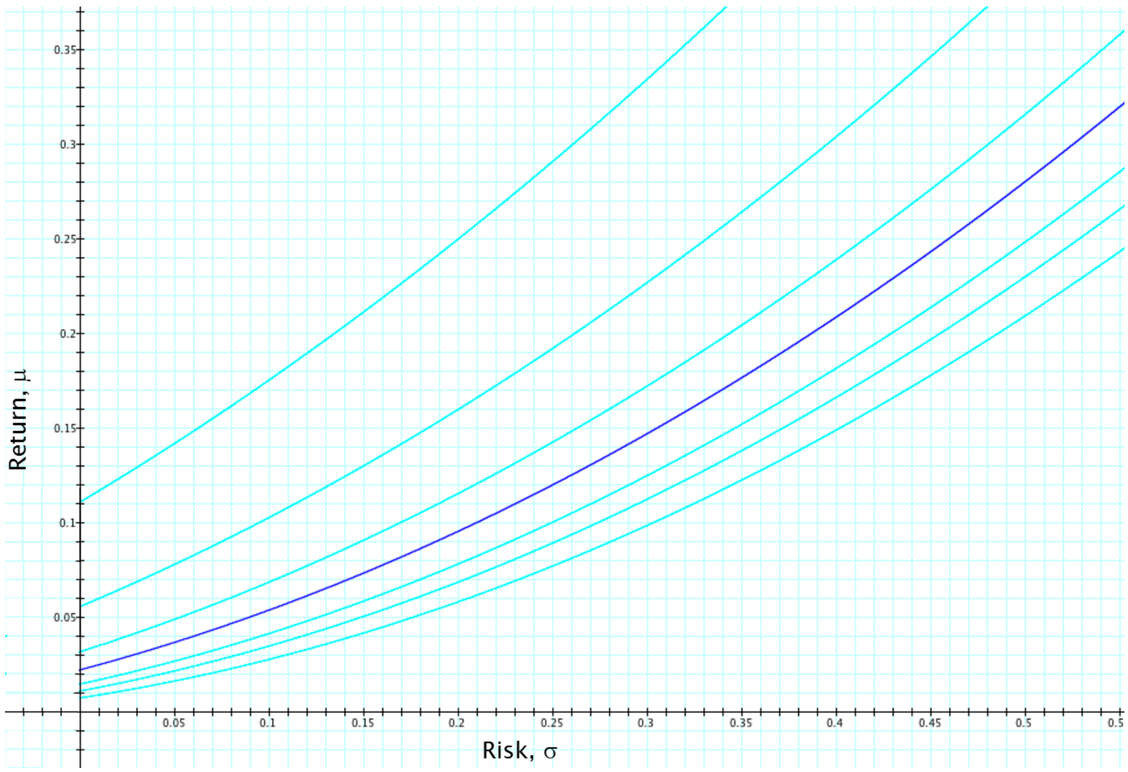

Figure 6: The Dependence of Goal Probability Level Curves on Investment Tenure. The top GPLC corresponds to an Investment Tenure of 2 years, instead of the base case value of 10 years. Below that are the GPLCs as the Investment Tenure is increased to 4 years, 7 years, 10 years (in dark blue for the base case), 15 years, 20 years, and 30 years. We note that the GPLCs decrease at a slower pace as the Investment Tenure increases. As in Figure 4, all parameters other than the Investment Tenure have their base case values, as given in Figure 3.

than the Loss Threshold Probability.

The Loss Threshold Wealth, of course, is a smaller number than the Target Wealth. Indeed, it may be below or equal to the Initial Wealth, as well as above it. The Loss Threshold Probability is always higher than the Target Probability, however, as a rule of thumb, values exceeding 99% may be problematic, because tail events above this level are hard to model accurately.

Mathematically, the Loss Threshold Curve (LTC) is characterized in the same way as the Goal Probability Level Curve. That is, the LTC is described by equation (2) with the Loss Threshold Wealth substituted for $W(t)$, and z_0 chosen so that the Loss Threshold Probability is equal to the probability that a normal random variable takes a value less than z_0. Satisfying the Loss Threshold restriction means using a portfolio whose standard deviation and expected value remain above the Loss Threshold Curve. That is, we would like to have a portfolio in the (σ, μ) plane that lies in the intersection of the Goal Region and the region on or above the Loss Threshold Curve.

4.2.3 Efficient Frontier

We consider a portfolio with access to n assets. To minimize the portfolio risk, σ, for a given return, μ, we ideally choose a combination of these n assets that puts us on **the Efficient Frontier,** defined by the formula:

$$\sigma = \sqrt{a\mu^2 + b\mu + c}, \tag{4}$$

which shows the relationship between σ and μ traces out a hyperbola (See Figure 7). The constants, a, b, and c are defined by $\mathbf{M}$, which is a vector of the n expected returns; $\mathbf{O}$, which is a vector of n ones; and $\boldsymbol{\Sigma}$, which is the covariance matrix of the n assets, via the following equations:

$$\begin{aligned} a &= \mathbf{h}^\top \boldsymbol{\Sigma} \mathbf{h} \\ b &= 2\mathbf{g}^\top \boldsymbol{\Sigma} \mathbf{h} \\ c &= \mathbf{g}^\top \boldsymbol{\Sigma} \mathbf{g}, \end{aligned}$$

where the vectors $\mathbf{g}$ and $\mathbf{h}$ are defined by

$$\begin{aligned} \mathbf{g} &= \frac{l\boldsymbol{\Sigma}^{-1}\mathbf{O} - \mathbf{k}\boldsymbol{\Sigma}^{-\mathbf{1}}\mathbf{M}}{lm - k^2} \\ \mathbf{h} &= \frac{m\boldsymbol{\Sigma}^{-1}\mathbf{M} - \mathbf{k}\boldsymbol{\Sigma}^{-\mathbf{1}}\mathbf{O}}{lm - k^2}, \end{aligned}$$

and the scalars k, l, and m are defined by

$$\begin{aligned} k &= \mathbf{M}^\top \boldsymbol{\Sigma}^{-\mathbf{1}} \mathbf{O} \\ l &= \mathbf{M}^\top \boldsymbol{\Sigma}^{-\mathbf{1}} \mathbf{M} \\ m &= \mathbf{O}^\top \boldsymbol{\Sigma}^{-\mathbf{1}} \mathbf{O}. \end{aligned}$$

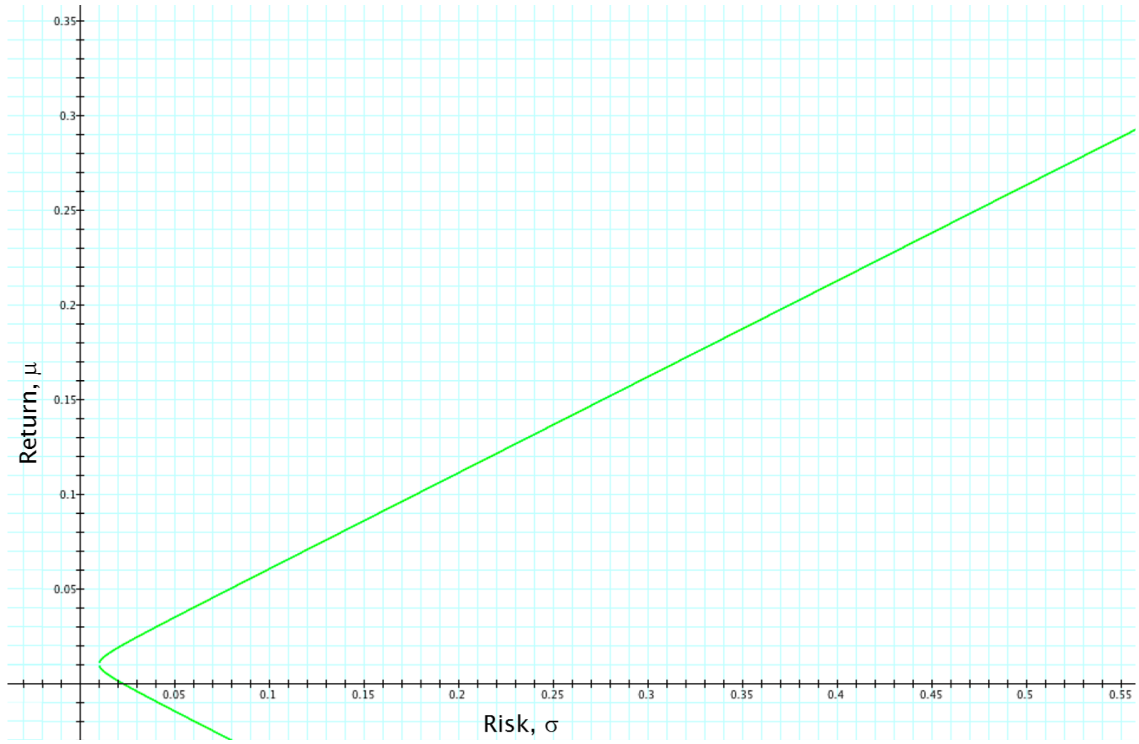

Figure 7: An Efficient Frontier Example. The hyperbola in the risk-return plane corresponding to an Efficient Frontier. This Frontier was based off of 10 years of data for three assets: a low risk money market fund, a medium risk stock market index, and a higher risk tech stock index. The full set of feasible portfolios lie on the Efficient Frontier or to the right of it, but only portfolios that lie on the upper half of the Efficient Frontier (e.g., on the part of the green curve where $\mu \geq 0.01$ in the figure) should be used. These are the portfolios where expected return is maximized for a fixed amount of portfolio risk.

Imposing additional restrictions like only holding long positions usually has only small effects on the Efficient Frontier, see Das, Markowitz, Scheid, and Statman (2010), although it may impose a maximum value beyond which the hyperbolic curve is cut off.

We will only consider portfolios that are on our (potentially restricted) Efficient Frontier. Ideally, they will also be in the Goal Region and on or above the Loss Threshold Curve (See Figure 8). Therefore, we ensure that optimal asset allocation is combined with goals-based portfolio optimization.

4.3 Building Blocks for a Goals-Based Strategy

We now combine the Goal Region, the Loss Threshold, and the Efficient Frontier into a specific goals-based strategy for investors that is mean-variance efficient.

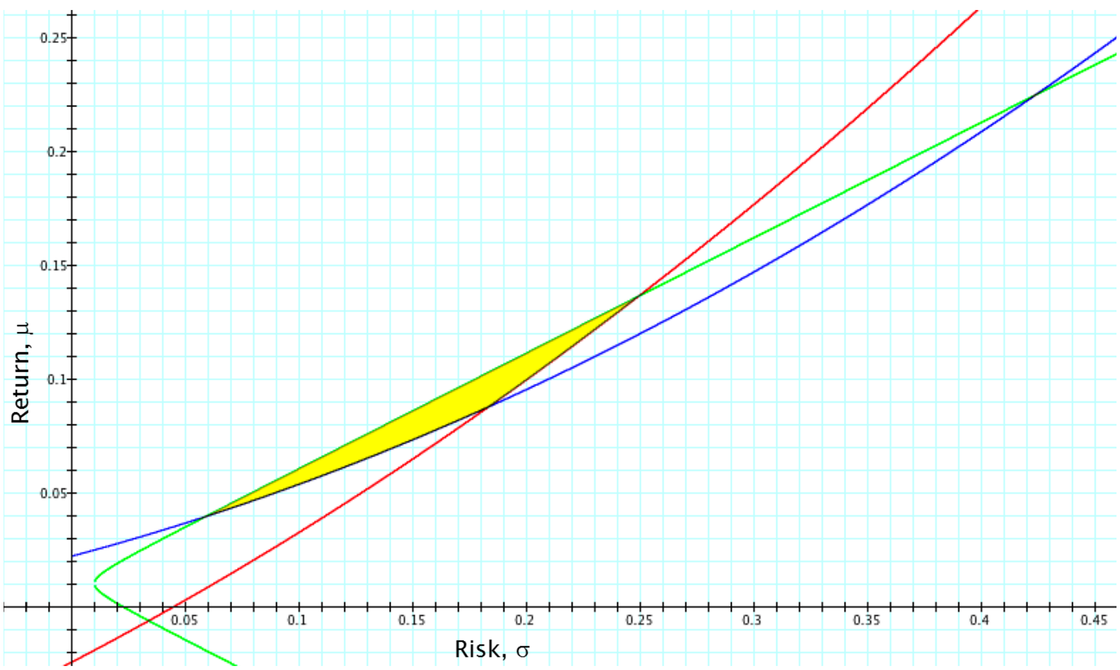

Figure 8: Feasible Portfolios that Satisfy the Investor's Target Goals and Loss Threshold. The golden region corresponds to portfolios that are feasible (since they are to the right of the Efficient Frontier in green), while also satisfying both the investor's goals (by staying above the boundary of the goal region in dark blue) and their loss threshold tolerance (by staying above the Loss Threshold Curve, shown in red). The red Loss Threshold Curve (LTC) in this figure corresponds to a 95% chance that the initial portfolio wealth of $400,000 will be worth at least $300,000 at the end of the Investment Tenure of 10 years. The Efficient Frontier is the frontier given in Figure 7. The Goal Region is the base case given in Figure 3.

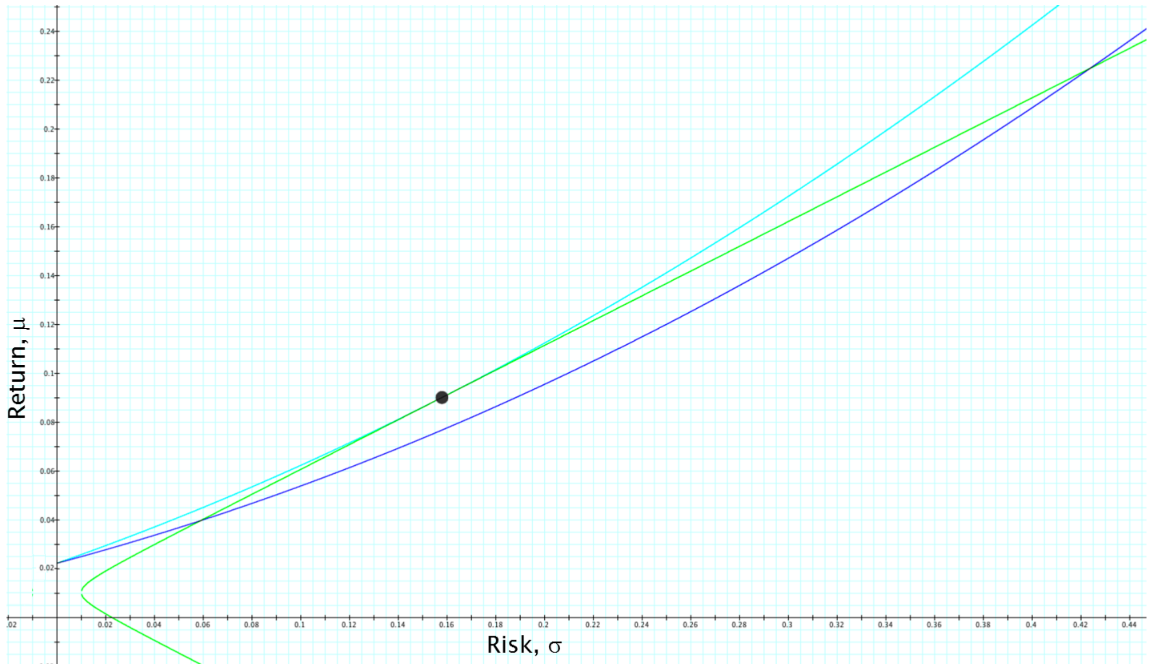

Figure 9: Optimizing the Probability of an Investor Meeting their Goal. The base case from Figure 3 corresponds to an 80% GPLC. We increase this percentage, as was done in Figure 4, until the GPLC intersects the Efficient Frontier at only a single tangent point. This single point is the Optimal Goal Probability Point, which corresponds to the highest percentage GPLC that is feasible for the investor. In this figure the Optimal Goal Probability Point is located at (0.158, 0.0902). This is the black point on the graph, which is on the 86.6% GPLC.

4.3.1 Relevant Points on the Efficient Frontier

Because Goal Probability Level Curves (GPLCs) are upward curving (convex) parabolas and the Efficient Frontier curves downward (i.e., is concave), the value of z can be adjusted to determine the unique GPLC that intersects the upper half of the Efficient Frontier (i.e., the half of the frontier where μ increases as σ increases) at exactly one location. We will call this unique (σ, μ) pair the **Optimal Goal Probability Point**. When the Optimal Goal Probability Point lies in the interior of the Efficient Frontier (as opposed to an endpoint of the frontier, which may exist if we prohibit short selling, for example), it corresponds to the point of tangency between the unique Goal Probability Level Curve and the Efficient Frontier. (See Figure 9). We show how to use the method of Lagrange multipliers to explicitly compute this point of tangency in Subsection 4.3.4.

In general the boundary of the Goal Region will, at least initially, intersect the Efficient Frontier in two locations, which we will call the **Lower Goal Point** and the **Upper Goal Point**. Should the boundary of the Goal Region dip so low that it goes below the left endpoint of the upper half of the Efficient Frontier, we set the Lower Goal Point to equal this left endpoint. A similar assignment is made for the Upper Goal Point if the boundary dips below the right endpoint of a restricted Efficient Frontier. If the Goal Region does not intersect the Efficient Frontier, the Lower and Upper Goal Points are not defined. When they are defined, the Optimal Goal Probability Point will always lie between them or on them.

Next, we define the **Loss Threshold Point** to be the point of intersection of the Loss Threshold Curve parabola with the upper half of the Efficient Frontier. In the less common case where the LTC intersects the upper half of the Efficient Frontier in two locations, the Loss Threshold Point is the rightmost intersection point. As time evolves, the LTC may lie completely above the Efficient Frontier, in which case, the value of z_0 in equation (2) for the LTC is reduced until the parabola again intersects the Efficient Frontier at a unique location, which is then defined as the Loss Threshold Point.

Investors must specify inputs that are at least initially feasible, which means the LTC must intersect the upper half of the Efficient Frontier and the Loss Threshold Point must be to the right of the Lower Goal Point. This means that we have a portfolio that lies on the Efficient Frontier that is in the Goal Region and on or above the LTC. This corresponds to a portfolio that lies on the part of the Efficient Frontier that forms the upper boundary of the golden region in Figure 8.

Finally, we define the **Halfway Point** between two points on the Efficient Frontier to be the point on the Frontier whose volatility, σ, is equal to the average of the two points' volatilities.

4.3.2 Goals-based Investing in Good States

Good states are defined by the existence of the golden region in Figure 8 and having the Loss Threshold Point be to the right of the Optimal Goal Probability Point, where "to the right" means having higher μ and σ values on the upper half of the Efficient Frontier. Recall that the investor had been asked for their investment preference in good states by selecting one of three options: Option 1, optimizing the probability of obtaining their goal; Option 2, optimizing their expected return subject to their other specifications, like maintaining their Target Probability; or Option 3, a mix between these two options.

In the best case scenario, the Loss Threshold Point is to the right of the Upper Goal Point. In this case Option 1 corresponds to selecting the portfolio that corresponds to the Optimal Goal Probability Point, Option 2 corresponds to selecting the Upper Goal Point, and Option 3, if a half and half mix is selected, corresponds to selecting the Halfway Point between Option 1 and Option 2 (See Figure 10).

In the next best case scenario, the Loss Threshold Point lies between the Optimal Goal Probability Point and the Upper Goal Point. In this case, for each of the three options, we use whichever is more to the left: the same portfolio selected in the best case scenario or the Loss Threshold Point. This enforces maintaining the Loss

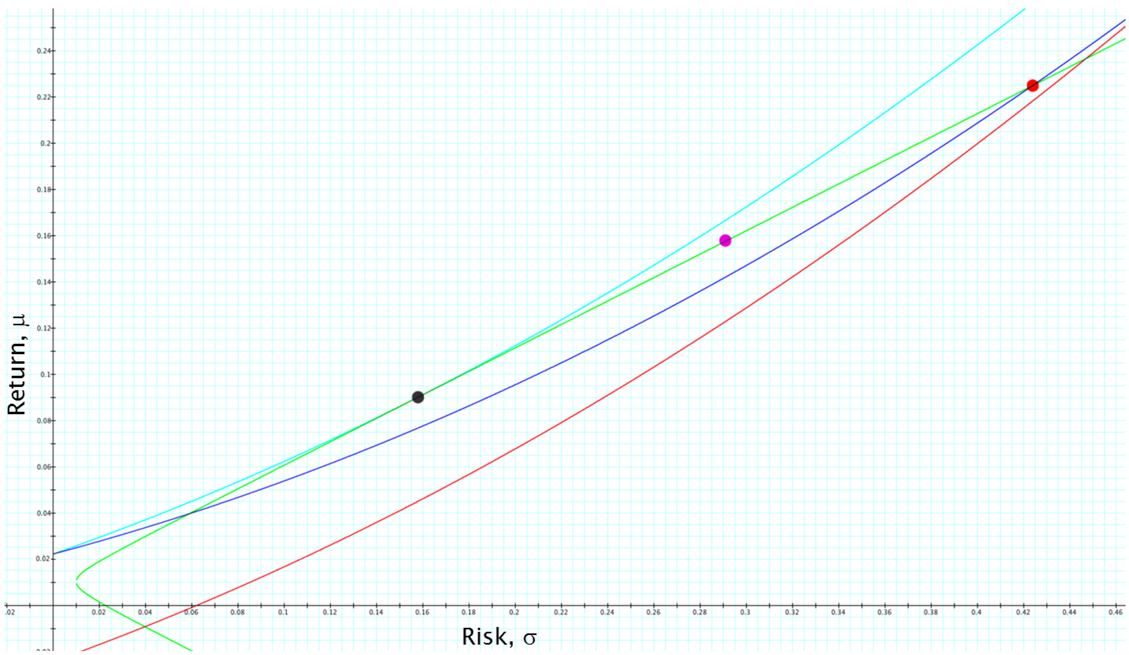

Figure 10: Options in the Best Case Scenario. Option 1 maximizes the investor's probability of achieving their goal. Option 2 maximizes expected return subject to maintaining the investor's other specifications, like their Target Probability. Option 3 corresponds to a point on the frontier between the first two options. Option 1, the optimal goal probability point, (0.158, 0.0902), is in black. Option 2, the upper goal point, (0.424, 0.225), is in red. An example of Option 3 is the halfway point, (0.291, 0.158), which is in magenta, where the risk, σ, is the average of the σ values for Option 1 and for Option 2.

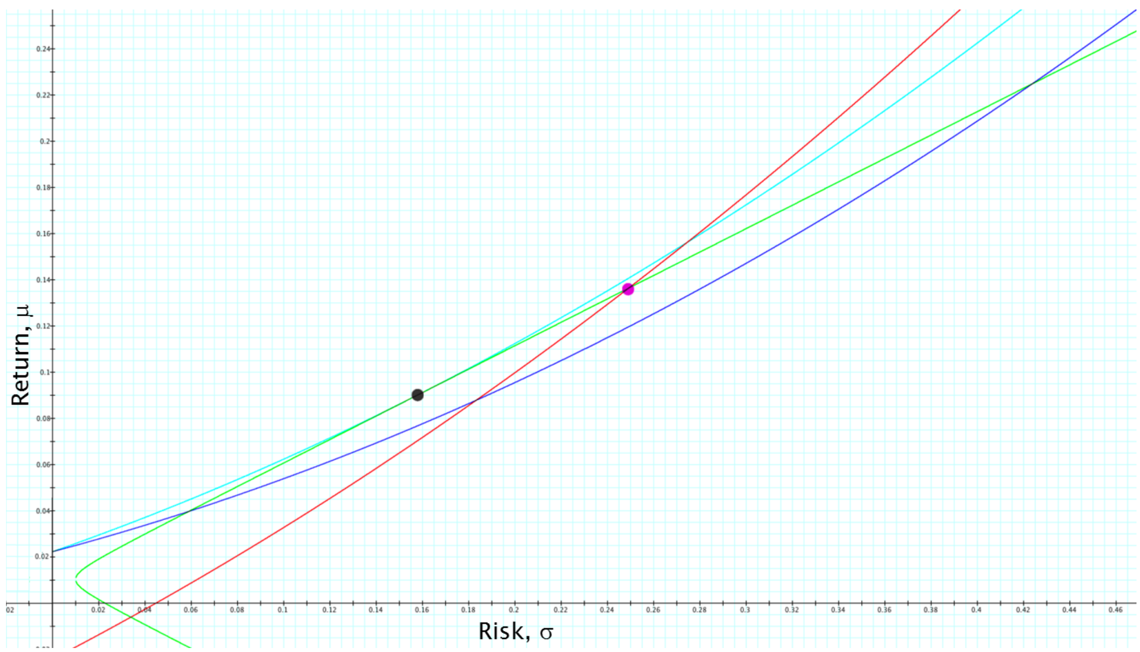

Figure 11: Options in the Next Best Case Scenario. To conform to the investor's loss threshold, the options from Figure 10 are restricted to remain on or above the red Loss Threshold Curve. Option 1, the optimal goal probability point, is still the black point at (0.158, 0.0902). Option 2 and Option 3, however, are restricted by the LTC, so both now correspond to the loss threshold point, (0.249, 0.136), which is in magenta.

Threshold (See Figure 11).

4.3.3 Goals-Based Investing in Bad States

When the Loss Threshold Point is to the left of the Optimal Goal Probability Point and/or the golden region in Figure 8 disappears, we are in a bad state. In bad states, the Loss Threshold Point may still be to the right of the Optimal Goal Probability Point. This will only happen if the Goal Region lies strictly above the Efficient Frontier, so the golden region no longer exists. In this case, the portfolio at the Optimal Goal Probability Point is selected.

When the Loss Threshold Point is to the left of the Optimal Goal Probability Point, we use the investment preference specified by the investor in bad states. Recall, this is one of three options: Option 1, optimizing the probability of obtaining their goal; Option 2, optimizing maintaining their Loss Threshold; or Option 3, a mix between these two options.

Option 1 corresponds to selecting the portfolio that corresponds to the Optimal Goal Probability Point, Option 2 corresponds to selecting the Loss Threshold Point, and Option 3, should a half and half mix be selected, corresponds to selecting the Halfway Point between Option 1 and Option 2. Note that for cases where the golden region in Figure 8 still exists, all three options select a portfolio within the golden region. These three options are also well defined in cases where the Loss Threshold and/or the Goal Region do not intersect the Efficient Frontier. By using the Optimal Goal Probability Point, we maximize the investor's likelihood of attaining their goal. By using the Loss Threshold point, we maximize the likelihood of not violating the investor's specified tolerance for underperformance (See Figure 12).

4.3.4 Computing the Optimal Goal Probability Point

Recall that to obtain the Optimal Goal Probability Point, we increase the value of z in equation (2) until the GPLC that corresponds to this value of z intersects the upper half of the Efficient Frontier in only a single location. This location is the Optimal Goal Probability Point.

To calculate the Optimal Goal Probability Point, we first rearrange equation (2) to isolate z:

$$z(\sigma, \mu) = \frac{1}{\sigma}\left(\left(\mu - \frac{\sigma^2}{2}\right)\sqrt{t} - \frac{1}{\sqrt{t}}\ln\left(\frac{W(t)}{W(0)}\right)\right). \tag{5}$$

We look to optimize z subject to the restriction that we remain on the Efficient Frontier given in equation (4), which is easily rearranged into the form

$$g(\sigma, \mu) = a\mu^2 + b\mu + c - \sigma^2 = 0. \tag{6}$$

This means employing the method of Lagrange multipliers, namely simultaneously solving

$$\nabla z(\sigma, \mu) = \lambda \nabla g(\sigma, \mu) \tag{7}$$

for some scalar Lagrange multiplier, λ, along with the Efficient Frontier restriction

$$g(\sigma, \mu) = 0.$$

From equation (7), using equations (5) and (6), we have from differentiating with respect to σ that

$$-\frac{1}{\sigma^2}\left(\mu\sqrt{t} - \frac{1}{\sqrt{t}}\ln\left(\frac{W(t)}{W(0)}\right)\right) - \frac{\sqrt{t}}{2} = -2\lambda\sigma$$

and from differentiating with respect to μ that

$$\frac{\sqrt{t}}{\sigma} = \lambda(2a\mu + b).$$

Combining these two equations so as to remove λ yields, after rearrangement,

$$2\sqrt{t}\sigma^2 = \left(\mu\sqrt{t} - \frac{1}{\sqrt{t}}\ln\left(\frac{W(t)}{W(0)}\right) + \frac{\sqrt{t}}{2}\sigma^2\right)(2a\mu + b).$$

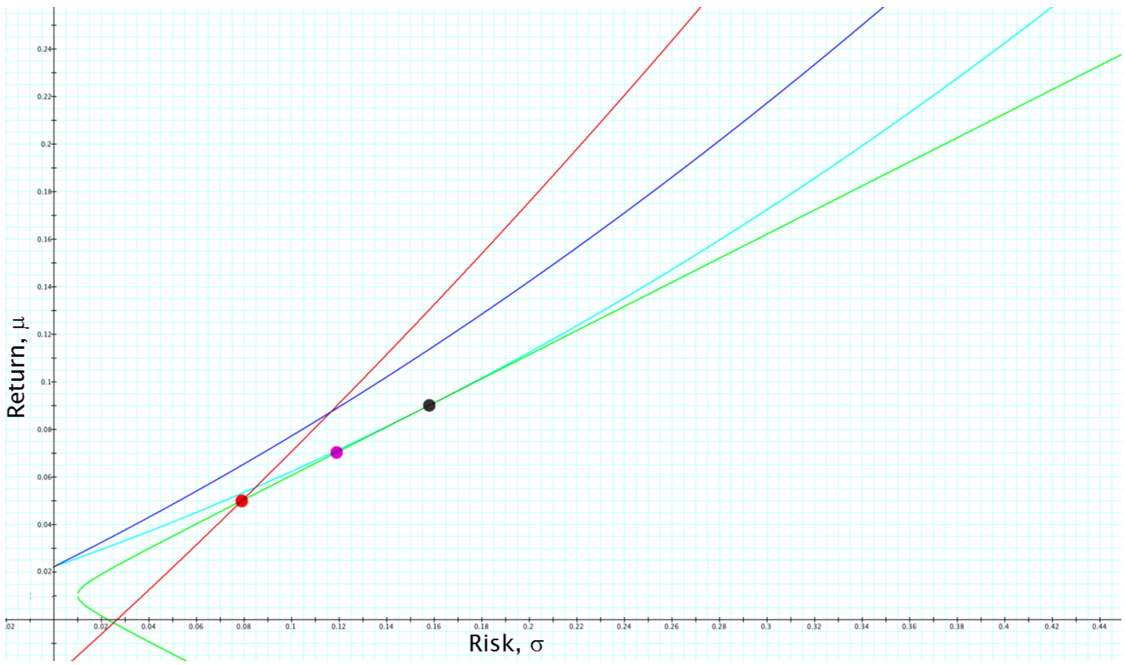

Figure 12: Options in Bad States. The Loss Threshold Point is now to the left of the Optimal Goal Probability Point and/or the golden region in Figure 8 no longer exists. In this figure, both are the case. The investor has selected one of three options: Option 1, the Optimal Goal Probability Point, (0.158, 0.0902), which is in black, corresponds to prioritizing attaining the investor's Target Wealth at the expense of ignoring their Loss Threshold. Note that the boundary of the goal region in this figure is above the efficient frontier, which means that the probability of attaining the Target Wealth, while optimized by Option 1, will still be below the Target Probability. Option 2, the Loss Threshold Point, (0.0791, 0.0500), which is in red, corresponds to prioritizing the investor's Loss Threshold at the expense of ignoring the investor's Target Wealth. In this case, the investor's Loss Threshold Probability can be preserved, because the LTC still intersects the Efficient Frontier. Should the LTC lie above the Efficient Frontier and therefore not intersect it, we can still minimize the probability that the investor will end up below their Loss Threshold Wealth, although this minimized probability will now be below the Loss Threshold Probability. As before, Option 3 corresponds to a point on the frontier between the first two options. Assuming an equal mix of the other two options, we get the halfway point, (0.119, 0.0704), in magenta, where the risk, σ, is the average of the σ values for Option 1 and for Option 2.

Now we use the Efficient Frontier restriction to substitute $a\mu^2+b\mu+c$ for σ^2, and, after rearrangement, we obtain a third degree polynomial equation, i.e., a cubic equation, for the value of μ:

$$c_3\mu^3 + c_2\mu^2 + c_1\mu + c_0 = 0, \tag{8}$$

where

$$\begin{aligned} c_3 &= a^2 \\ c_2 &= \frac{3ab}{2} \\ c_1 &= ac + \frac{b^2}{2} - b - \frac{2a}{t}\ln\left(\frac{W(t)}{W(0)}\right) \\ c_0 &= \frac{bc}{2} - 2c - \frac{b}{t}\ln\left(\frac{W(t)}{W(0)}\right). \end{aligned}$$

In general, cubic equations (8) have either one real (and two complex) roots or three real roots. From the geometry of equation (7), each of the real roots in equation (8) must correspond to locations where a GPLC and the Efficient Frontier are tangent to each other. Since GPLCs are convex parabolas, when there is only one real root, it must correspond to a point on the upper half of the Efficient Frontier. Note that the upper half of the Efficient Frontier corresponds to the region where $\mu \geq -\frac{b}{2a}$, since the Efficient Frontier from equation (4) can be rewritten in the form

$$\sigma = \sqrt{a\left(\mu + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c}.$$

When there are three real roots, there must still be one root corresponding to a point in the upper half of the frontier. The other two roots (which may both form a double root) must always correspond to points (or a point in the case of a double root) on the lower half of the Efficient Frontier (See Figure 13).

Therefore, there will always be exactly one root of our cubic equation where $\mu \geq -\frac{b}{2a}$. This root, which may be found exactly using Cardano's formula for cubic equations or approximated using a numerical solver, must be the μ coordinate of the Optimal Goal Probability Point, and the corresponding σ coordinate is then determined from equation (4): $\sigma = \sqrt{a\mu^2 + b\mu + c}$.

4.4 Observations Concerning Our Goals-based Investing Approach

The main new perspective and contribution in our algorithm is that it quantifies the probability of attaining an investor's goal—be it growing the worth of a portfolio above a Target Wealth or staying above a Loss Threshold—and clarifies the role these probabilities can play in a GBWM portfolio. We can not only generate the

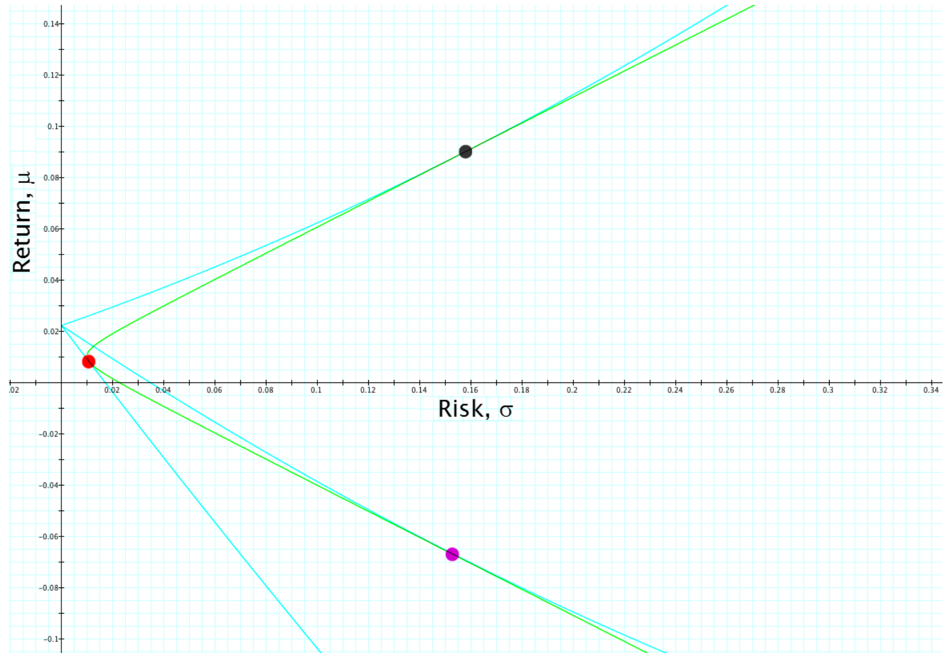

Figure 13: Three points corresponding to three real roots of the cubic equation (8) for μ. The three curves in light blue are the only GPLCs that share a tangent line with the Efficient Frontier at their point of intersection with the frontier. The highest tangential intersection point, (0.158, 0.0902), in black, is the only tangential intersection point on the upper half of the Efficient Frontier, meaning the half of the frontier where μ increases as σ increases. There will always be exactly one tangential intersection point on the upper half of the Efficient Frontier, and this point is the Optimal Goal Probability Point. This point is on the GPLC that intersects the Efficient Frontier with the highest probability level. The other two tangential intersection points—here (0.0108, 0.00829), in red, and (0.153, -0.0669), in magenta—either both exist, possibly as a double root, or neither exists. When they both exist, they will always lie on the lower, irrelevant half of the Efficient Frontier, like we see here.

range of portfolios that satisfy the investor's specified probabilities for ending above their Target Wealth and for ending above their Loss Threshold Wealth (as in Figure 14), but also, we can generate the specific portfolio that, in addition, best satisfies the investor's preferences under both good states and bad states.

Our analysis allows us to determine the answers to some questions that may not be intuitively clear. For example, it is intuitive to advisors that an investor choosing a portfolio with too little risk and return will have a small probability of attaining their financial goals. It is also clear that increasing an investor's risk and return increases the potential for larger losses when losses occur. But it is less clear whether or not increasing risk and return increases or decreases the actual probability of the investor attaining their goals. Our analysis clearly answers this question: increasing an investor's risk and return may initially increase the probability of attaining the

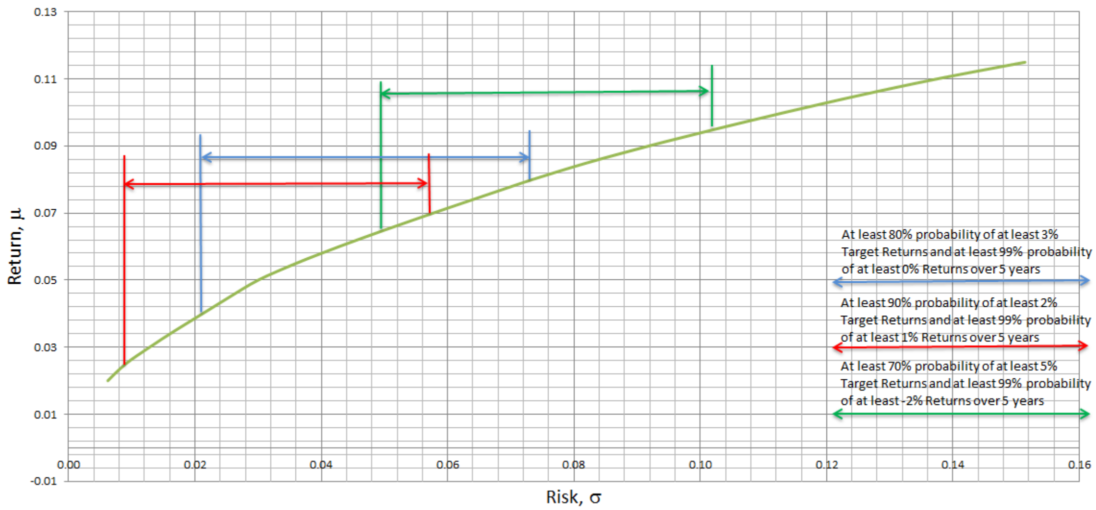

Figure 14: Ranges of Portfolios on the Efficient Frontier that Satisfy Different Combinations of Target Goals and Loss Thresholds. The ranges on the Efficient Frontier here correspond to the upper boundary of the golden region shown in Figure 8 under three different circumstances. Note that we specify rates of return, r, in this figure in place of the Initial Wealth, $W(0)$, and the Target Wealth/Loss Threshold Wealth, $W(t)$, where $r = \frac{1}{t}\ln\left(\frac{W(t)}{W(0)}\right)$.

investor's goal, but further increases will eventually diminish this probability to zero. And, of course, our analysis goes further by finding the specific risk/return combination on the Efficient Frontier that optimizes the probability of the investor attaining their Target Wealth, as well as the interval on the frontier between the Lower and Upper Goal Points that will attain the Target Wealth with at least the investor's specified Target Wealth Probability.

Note also that the Lower Goal Point is not relevant to any advice given to the investor. The Upper Goal Point is important because it tells us how much an investor is willing to increase the risk of not attaining their goal so that they can increase their expected gains, but when the Loss Threshold is not relevant, the algorithm will not choose a portfolio between the Lower Goal Point and the Optimal Goal Probability Point. This is because switching from the Optimal Goal Probability Point to a portfolio between the Lower Goal Point and the Optimal Goal Probability Point increases, as opposed to decreases, the risk of the investor not attaining their Target Wealth goal, and, at the same time, it decreases the investor's expected gains.

It is well established that most investors who do not consult financial advisors will sell in a bear market and buy in a bull market, despite the fact that they are "buying high and selling low." Automatic rebalancing, including using Life Cycle funds, can help ameliorate this by automatically doing some buying in bear markets and selling in bull markets as it rebalances to its fixed percentages of less risky and more risky asset classes.

Does our approach automatically "buy low and sell high" as any investment strategy should? The answer, generally, is yes, and to a more significant degree than automatic rebalancing and Life Cycle funds do. For example, consider the case of a bear market where the poor market conditions lead to a significant decrease in the portfolio's worth. This corresponds to $W(0)$ being reduced, which increases the μ-intercept given in equation (3) for both GPLCs and LTCs and, in fact, creates a corresponding upwards translation of the entire GPLC or LTC by this amount, as shown in Figure 5. If, for bad states, the investor has selected Option 1, this will induce a shift to higher risk-return portfolios on the Efficient Frontier. That is, the investor will be buying more risky stocks in a bear market, as they should. We note that this is better than just staying at the same point on the risk-return plane, as automatic rebalancing or Life Cycle funds do.

On the other hand, if the investor has selected Option 2 for bad states, this will correspond to moving to lower risk-return portfolios as long as the LTC continues to intersect the Efficient Frontier. But should performance continue to worsen to the point that the LTC lies strictly above the Efficient Frontier, the selected portfolio will then change course and begin to move to higher risk-return locations. This makes sense given the investor's investment preference: in a slight downmarket, the investor can protect themselves from loss by moving away from risk, but in a significant downmarket, the investor will switch to purchase more risky, but now even more inexpensive, assets to optimize their probability of not ending below their loss threshold.

There is extensive discussion of GBWM in the practitioner literature, see for example Chabbra (2005). Our paper complements this by providing a systematic mathematical exposition of the approach. Next, we comment on some practical considerations.

4.5 Running Our Approach in Practice

Working with the Efficient Frontier has advantages from a practical standpoint, other than the main property that it minimizes portfolio volatility. Some advisors strongly prefer to work with a small number of funds, others with a large number. Using the Efficient Frontier can accommodate any number of funds. Some companies use historical data to determine the mean, standard deviation, and correlations between funds. Others use projections based on current data or a combination of historical data and future projections. All of these can be accommodated. This is a strong point of the GBWM framework, i.e., that it is done in two stages. First, we pick a set of efficient portfolios that may be acceptable to an investor in terms of their portfolio weights and their distributional properties, and any other constraints that the investors may require. Second, from this set, we then choose the portfolio that best meets the investor's goals. Operationally also, this has the advantage that the two steps may be performed by different teams in a wealth management firm, both applying their specific expertise.

Also, our model is flexible in two ways. First, although we have used geometric Brownian motion to model returns in this paper, there is no problem with adapting our algorithm to a preferred alternative model for returns. The only alteration to the algorithm that is needed is in the computation of the Optimal Goal Probability Point, which would generally need to be located numerically, instead of through the use of Lagrange multipliers. Second, we may overlay options positions on the portfolio as well, as these may be appropriate securities with which to meet the Goal Probability and Loss Threshold constraints.[7]

Although the discussion of our algorithm has centered on its ability to determine the probability of attaining a Target Wealth or maintaining a Loss Threshold Wealth, it is important to note that our algorithm can also determine a complete probability distribution for the investor's returns at any future time. That is, along with a recommended portfolio for the investor, a probability distribution for the wealth of the portfolio could also easily be generated for, say, every year until the end of the Investment Tenure.

Once the eight pieces of information listed at the end of Section 3 are gathered from the investor, our algorithm can be run once and the portfolio can be maintained with what our algorithm recommends, but we strongly suggest instead that the algorithm should be rerun periodically as the conditions in the market, and therefore in the Efficient Frontier, change. This can be done automatically at regular intervals for investors who prefer a hands-off approach. In addition, for hands-on investors, it can be done manually whenever the investor desires, including when the investor wants to change their goals or consider the effect of additional investments or withdrawals. The key thing to note is that even though our algorithm is static, it can be updated dynamically, either to adapt to new market data or a change in the eight pieces of information the investor supplies to the algorithm.

5 Conclusion

This paper presents a new approach to goals-based wealth management (GBWM) that focuses on understanding risk as the probability of not attaining the investor's goals, not just the traditional perspective of risk being the standard deviation of the investor's portfolio. Our GBWM approach is a framework that recognizes investor

[7]We note, however, that put and call options, as well as futures contracts and most other derivatives, are prohibited in 401(k) plans. They are also prohibited by many IRA custodians. The sole exception to this is covered calls, which may be purchasable through a 401(k) brokerage link or through an IRA. (See the Investopedia articles http://tinyurl.com/y797jqpq and http://tinyurl.com/ycdust9u.) In accounts that are intended for the long term goal of retirement where options are available, the fact that options usually have short maturities creates rollover risk, as discussed in Chabbra (2005).

goals and offers a mathematical foundation for analyzing portfolios, while remaining fully consistent with modern portfolio theory. It also satisfies nine desirable GBWM Properties listed in Section 3 that are generally not fully satisfied in extant portfolio strategies.

Our approach has significant advantages for both advisors and their clients. It enables advisors to give individualized advice (GBWM Property 2) after ascertaining the goals of the investor in a manner that is clearer (GBWM Property 1), because it uses language and ideas that market research has shown are more natural and understandable to investors (GBWM Property 5) than the language that advisors typically currently use. More specifically, the advisor works with the client to determine eight pieces of information: the size of their initial investment, their time frame, their goal wealth at the end of this time frame, and the acceptable probability of attaining this goal wealth, as well as a loss threshold wealth and an acceptable probability of not going below this loss threshold wealth.

For the final two pieces of information, the investor is asked to choose 1) in good states where the portfolio is well funded, how much the investor prioritizes decreasing the risk of not attaining their goal wealth versus increasing their expected returns, and 2) in bad states where the portfolio is less well funded or has become underfunded, how much the investor prioritizes decreasing the risk of not attaining their goal wealth versus decreasing the risk of ending below their loss threshold. For good states, since risk is defined in terms of not attaining their goal instead of the portfolio standard deviation, the meaning of this prioritization decision is more clear for the investor. For bad states, the investor is being asked to prioritize between their dream of attaining their goal wealth and their fear of ending below their loss threshold wealth, which, again, involves more intuitive notions for the investor, and makes for interactions of greater clarity between advisors and their clients.

Using just these eight items of goals-based information from the investor, this paper details an algorithm that determines a portfolio that best fits the investor's individual specifications. This portfolio is formed by determining and analyzing the parabolic geometry in the risk-return plane that corresponds to any specific probability for attaining a goal, and then considering the intersection of these parabolas with the hyperbolic geometry that corresponds to the Efficient Frontier. This produces a portfolio on the Efficient Frontier (GBWM Property 7), thereby minimizing unnecessary standard deviation risk, while also being able to calculate and minimize the risk of not attaining the investor's goals (GBWM Properties 3 and 4). The algorithm's portfolio recommendation is based on the overall state of the portfolio, as opposed to the performance of the individual portfolio components (GBWM Property 6). Further, the portfolio can be recalculated and rebalanced either automatically or manually at any time (GBWM Property 9) to adjust to changes in the market or changes the investor wishes to make to the eight pieces of information they have specified (GBWM Property 8).

Employing our approach allows advisors to help their clients in a number of ways that alternatives like Life Cycle funds, which exhibit almost none of these nine GBWM properties, are not designed to do. Further, because the required input is so simple and the algorithm is designed to work in both good and bad states, it can be used effectively both for clients that are wealthy and clients that are not wealthy.

We mention four future directions for improving and expanding the GBWM approach explained in this paper:

- Multiple goals: In this paper, we have restricted our analysis to a single investment with one dream-oriented goal and one fear-oriented goal in the same time frame. In reality, of course, investors look at multiple time frames that correspond to different goals like buying a house, sending children to college, and funding retirement.

- Periodic investments and withdrawals: In this paper, we have restricted our analysis to a single initial investment and a single time frame for withdrawal. Ideally, we also want to model the effect of periodic investments, say, every year or with every paycheck, as well as periodic withdrawals to model common events like income withdrawal throughout retirement.

- Dynamic probability determination: In this paper, we have a static model that can be dynamically updated, but it would be better to have a dynamic model that would allow us to determine, or at least approximate via simulation, the effects on forecasted probabilities that future portfolio reconfigurations would induce. Under such dynamic models, it may also be possible to employ dynamic programming approaches to help determine new optimal portfolio strategies. This is the focus of a follow-up paper.

- Dynamic asset opportunity space: In this paper, we have assumed that the investable universe is stationary, i.e., the parameters of the processes driving the portfolio assets remain the same throughout the life of the goals-based portfolio. Generalizing to non-stationary asset processes can be accomplished in many ways, such as using multivariate GARCH models to model a changing return covariance matrix, coupled with a changing mean return vector based on a stochastic equity premium. A dynamic asset space may also be modeled with a regime-switching model.

In conclusion, we have developed a new geometry for goals-based wealth management that has several new, appealing properties, and aims to minimize the risk of investors failing to meet their goals, while remaining fully consistent with mean-variance portfolio theory. We believe that the new approach presented here will free advisors to be able to respond to the goals expressed by their clients in a more concrete, direct way than has been available previously, improving both the relationship between advisors and their clients and the quality of the recommendations advisors provide to their clients.

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