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Solutions of Hamilton–Jacobi Equations and Scalar Conservation Laws with Discontinuous Space–Time Dependence

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We establish a unique stable solution to the Hamilton–Jacobi equation

$$u_t + H(K(x,t), u_x) = 0, \qquad x \in (-\infty, \infty), \qquad t \in [0, \infty)$$

with Lipschitz initial condition, where $K(x,t)$ is allowed to be discontinuous in the (x,t) plane along a finite number of (possibly intersecting) curves parameterized by t. We assume that for fixed k, $H(k,p)$ is convex in p and $\lim_{p\to\pm\infty}|\frac{H(k,p)}{p}| = \infty$. The solution is determined by showing that if K is made smooth by convolving K in the x direction with the standard mollifier, then the control theory representation of the viscosity solution to the resulting Hamilton–Jacobi equation must converge uniformly as the mollification decreases to a Lipschitz continuous solution with an explicit control theory representation. This also defines the unique stable solution to the corresponding scalar conservation law

$$u_t + (f(K(x,t), u))_x = 0, \qquad x \in (-\infty, \infty), \qquad t \in [0, \infty)$$

with K discontinuous.

1. INTRODUCTION

In this paper we construct a unique stable solution, $u(x,t)$, to Hamilton–Jacobi equations of the form

$$u_t + H(K(x,t), u_x) = 0, \qquad x \in (-\infty, \infty), \qquad t \in [0, \infty),$$
$$u(x,0) = g(x), \tag{1}$$

where H is convex in u_x and has superlinear growth as $u_x \to \pm\infty$, $g(x)$ is Lipschitz continuous, and K is discontinuous along a finite number of curves in the (x,t) plane. By differentiating (1) with respect to x and defining $v = u_x$,

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we see that (1) is equivalent to the scalar conservation law

$$v_t + (f(K(x,t),v))_x = 0, \qquad x \in (-\infty,\infty), \qquad t \in [0,\infty),$$
$$v(x,0) = v^0(x) \equiv g'(x), \tag{2}$$

where f, the flux function, is the same function as H, the Hamiltonian.

Equations of the form (1) or (2) where K is discontinuous describe the behavior of many different physical phenomena including traffic flow [20], two-phase flow in porous media [6,7], and continuous sedimentation [5]. Equation (2) also arises in decoupled systems of conservation laws such as

$$\begin{aligned} v_{1,t} + (f_1(x,t,v_1))_x &= 0, \qquad x \in (-\infty,\infty), \qquad t \in [0,\infty),\\ v_{2,t} + (f_2(v_1,v_2))_x &= 0,\\ v_1(x,0) &= v_1^0(x),\\ v_2(x,0) &= v_2^0(x) \end{aligned} \tag{3}$$

when the solution to v_1—which is determined from the first equation and initial condition—has a finite number of shocks. We also note for the case of non-convex Hamiltonians/fluxes with discontinuous K, specific equations describing polymer flow and shape from shading using synthetic aperature radar have been studied in [14,17], respectively.

When $K(x,t)$ is continuous, (1) generally has a unique viscosity solution [2, 3] which corresponds to the entropy solution for (2), the associated conservation law. This uniqueness is attained by restricting the allowable behavior at shocks, which are discontinuities in u_x that can form in the solution for (1) even if g is smooth (and that can form in v, the solution for (2), even if v^0 is continuous). Besides requiring continuity over a shock for u in (1) (which corresponds to the Rankine–Hugoniot conditions for v in (2)), uniqueness is attained by only allowing characteristic curves to enter into a shock; that is, characteristics can never emanate (i.e. evolve) *from* a shock in the solution.

In contrast, when $K(x,t)$ is discontinuous, Lyons noted in 1983 that "The problem has properties of non-uniqueness which have not been previously encountered. The classical theory of conservation laws...is insufficient for our study...leaving our solution undetermined." [16]. Specifically, Lyons notes that prohibiting characteristic curves from emanating out of shocks where $K(x,t)$ is discontinuous is too strong a condition as it often would imply that no solution exists; however, once characteristic curves are allowed to evolve from shocks, there are immediate questions about how to restrict this behavior so that uniqueness is not lost. Resolving the question of how information is allowed to propagate through the characteristic curves where $K(x,t)$ is discontinuous is at the heart of the question of how to uniquely solve (1) and (2).

We next review specific cases of (1) and (2) where both uniqueness and existence have been attained.

For the linear case

$$u_t + K(x,t)u_x = 0, \qquad x \in (-\infty, \infty), \qquad t \in [0, \infty),$$
$$u(x,0) = g(x), \tag{4}$$

unique solutions of bounded variation (in x) have been obtained by Bouchut and James [1] and by Petrova and Popov [19] when K conforms to a one-sided Lipschitz condition and $g(x)$ is of bounded variation. Further, even when $g(x)$ is smooth, this bounded variation solution is, in general, not continuous. Numerical schemes for the linear case are discussed by Gosse and James [8].

Viewing the problem from a Hamilton–Jacobi perspective, Ishii slightly modified the standard definition of the viscosity solution for Hamilton–Jacobi equations and used his definition to establish existence and uniqueness for

$$u_t + H(x,t,u_x) = 0, \qquad x \in (-\infty, \infty), \qquad t \in [0, \infty),$$
$$u(x,0) = g(x),$$

where g is bounded and uniformly continuous and H is uniformly continuous in x and u_x, but only integrable (and therefore possibly discontinuous) in t [10]. We note that in this case characteristics must cross the lines of discontinuity so the issues arising from characteristics traveling along curves of discontinuity do not occur.

Viewing the problem from a conservation law perspective, Klingenberg and Risebro [13] determined a unique solution for

$$v_t + (K(x)f(v))_x = 0, \qquad x \in (-\infty, \infty), \qquad t \in [0, \infty),$$
$$v(x,0) = v^0(x), \tag{5}$$

where K has bounded total variation and is bounded away from zero, f is strictly convex and there exist constants a and b, where $a \leqslant v^0(x) \leqslant b$ and $f(a) = f(b) = 0$. Their method involved re-expressing (5) as a system of two equations (the second equation being $(K(x))_t = 0$) as was done by Temple and Isaacson [9] and applying front tracking, thereby also yielding a numerical scheme for their solution. In [11,12], Klausen and Risebro discuss extending the methods and results in [13] to equations of the form

$$v_t + (f(K(x), v))_x = 0, \qquad x \in (-\infty, \infty), \qquad t \in [0, \infty),$$
$$v(x,0) = v^0(x),$$

where (1) f is Lipschitz and smooth in K and v and strictly convex in v; (2) there are continuous functions $v_\alpha < v_T < v_\beta$, where $f(K, v_\alpha(K)) = f(K, v_\beta(K)) = 0$, $f_v(K, v_T(K)) = 0$, and $f(K, v_T(K))$ is bounded below zero; (3) $f(k, v)$ has no extrema on the interior of the region $(k, v) \in I \times [v_\alpha(K), v_\beta(K)]$, where I is the interval $[\inf_x K(x), \sup_x K(x)]$; (4) the initial condition $(K(x), v^0(x)) \in I \times [v_\alpha(K), v_\beta(K)]$ for all x; and (5) K has bounded variation, is discontinuous at a finite number of values, and its derivative exists and is bounded at all points of continuity. They also show that their solution is *stable*; specifically, they show that their solution must coincide with the limit as $\varepsilon \to 0$ of the solutions to

$$v_t^\varepsilon + (f(K^\varepsilon(x), v^\varepsilon))_x = 0, \qquad x \in (-\infty, \infty), \qquad t \in [0, \infty),$$
$$v^\varepsilon(x, 0) = v^0(x),$$

where $K^\varepsilon(x)$ is the convolution of $K(x)$ with the standard mollifier as defined in (8) in the next section.

In this paper we consider a stability criterion partially motivated by a related criterion we applied to the discontinuous eikonal equation in [18]. Our stability criterion here ends up being completely analogous to the Klausen and Risebro stability criterion; namely, we consider the limit as $\varepsilon \to 0$ of the solutions to

$$u_t^\varepsilon + H(K^\varepsilon(x, t), u_x^\varepsilon) = 0, \qquad x \in (-\infty, \infty), \qquad t \in [0, \infty),$$
$$u^\varepsilon(x, 0) = g(x), \tag{6}$$

where $K^\varepsilon(x, t)$ is, again, the smoothed version of the discontinuous function $K(x, t)$ formed by convolving K with the standard mollifier in the x direction. (In fact, while we use convolutions with the standard mollifier for simplicity in this paper, our proof will be easily extendable to a wider class of less standard mollifications which are specified in the next section after the standard mollifier is introduced in (8).) We note that both the use of the K function—as opposed to using $H(x, t, u_x)$—and the mollification of K in the x direction are *physically* motivated. In applications, K represents a discontinuous physical quantity, such as density, that should be stable to slight perturbations in space. Further, discontinuities in this quantity are really modeling (or at least can be modeled by) extremely quick continuous transitions from one state to another on the microscopic level. For example in (3), the system of conservation laws, the function v_1 is only discontinuous because the viscosity is assumed to equal zero in the model; in reality, of course, the viscosity may be extremely small but never zero so v_1 is really a macroscopic, discontinuous model for a quantity that is actually continuous, although it contains sharp transitions.

Our approach to our stability problem will be completely different from the approach of Klausen, Klingenberg, and Risebro, and it will allow us to accommodate a larger class of f and K functions. We will not convert (1) to a system of equations nor will we even take a conservation law perspective of the problem. Instead, we will view (1) from a Hamilton–Jacobi/viscosity solutions perspective, and, in particular, exploit the fact that, for any specific $\varepsilon > 0$, the unique viscosity solution for (6) has a control theory representation [15]. We will show that as $\varepsilon \to 0$, $u^\varepsilon(x,t)$ must converge uniformly to a Lipschitz continuous function $u(x,t)$ which we will define to be the unique stable solution to (1). We will also be able to express this solution by an explicit control theory representation.

Our assumptions for establishing this unique stable solution, $u(x,t)$, exists are (1) $H(k, p)$ is locally Lipschitz continuous and, for fixed k, is convex in p, (2) g is Lipschitz continuous, (3) $K(x,t)$ is bounded, (4) $K(x,t)$ is Lipschitz continuous except on a finite number of curves, where $K(x,t)$ can be discontinuous, (5) these curves are parameterized by t, have bounded speed, can intersect each other a finite number of times, and can fail to be smooth at a finite number of locations, (6) the characteristic speeds of the solution to (6) are assumed to be bounded uniformly in ε, and (7) for every k between $\inf_{(x,t)} K(x,t)$ and $\sup_{(x,t)} K(x,t)$, we have superlinear growth; that is,

$$\lim_{p \to \pm\infty} \left| \frac{H(k, p)}{p} \right| = \infty. \tag{7}$$

This superlinear growth condition, combined with the uniform bound on the characteristic speeds, implies that u_x^ε is bounded uniformly in ε as will be discussed further in the next section. For the moment we point out that for the case where the Hamiltonian takes the form

$$H(K(x,t), u_x) = K(x,t)|u_x|^n,$$

the superlinear growth condition implies that $n > 1$ and K is positive and bounded away from zero. This condition is sharp in the sense that when $n = 1$, we have the linear equation where, as stated earlier, numerous cases exist where u cannot be continuous, which, in fact, is due to u_x^ε becoming unbounded as $\varepsilon \to 0$. Further, if K is allowed to be both positive and negative, cases can again be constructed where u_x^ε becomes unbounded as $\varepsilon \to 0$ and u cannot be continuous. We emphasize that the purpose of condition (7) is to bound u_x^ε, so if the u_x^ε are bounded for another reason then condition (7) is unnecessary. Again, this will be explained further in the next section.

The control theory representation for our solution provides a mechanism for understanding how information can propagate through the curves where K is discontinuous by showing how to assign values for K to the cost function at these curves. One can use this representation to show that if H is monotone or convex in K for any fixed u_x (as is the case in (5)), then the value assigned to K at locations where K is discontinuous is either the limit of K as x increases to the discontinuity or the limit of K as x decreases to the discontinuity. This fact can be exploited—although we will not rigorously establish or further discuss it here—to show that monotone numerical methods that converge if K is continuous can also be employed to determine the solution when K is discontinuous. This is not necessarily the case when H is neither monotone nor convex in K. In this case, the value assigned to K at locations of discontinuity can be a value strictly between the limit of K as x increases to the discontinuity and the limit of K as x decreases to the discontinuity. For this case, the monotone numerical methods that converge if K is continuous will, in general, not produce the correct solution when K is discontinuous.

The organization of this paper is as follows: In the next section we discuss the above assumptions in more technical detail and review the control theory representation for the viscosity solutions to (6). In Section 3 we state the control theory representation for u, our solution to (1). In Section 4 we prove the pointwise convergence of $u^\varepsilon \to u$, by first discussing the basic skeleton and intuition behind the proof, then making some useful definitions, and finally explicitly establishing that $\liminf_{\varepsilon\to 0} u^\varepsilon \geqslant u$ and then that $\limsup_{\varepsilon\to 0} u^\varepsilon \leqslant u$. We conclude in Section 5 by showing that $u^\varepsilon \to u$ uniformly and that u is Lipschitz continuous.

2. CONTROL THEORY FORMULATION AND ASSUMPTIONS

We will work with the Hamilton–Jacobi form of our problem:

$$u_t + H(K(x,t), u_x) = 0, \qquad x \in (-\infty, \infty), \qquad t \in [0, \infty),$$
$$u(x,0) = g(x).$$

The Hamiltonian function, H, is assumed to be locally Lipschitz in both of its arguments and convex in u_x. The initial condition, $g(x)$, is assumed to be Lipschitz continuous. The function K is assumed to be bounded and we will use $K^{\inf} \equiv \inf_{(x,t)} K(x,t)$ and $K^{\sup} \equiv \sup_{(x,t)} K(x,t)$ to denote the bounds on K.

We define K^ε, the convolutions of K (that is, mollifications), in the standard way:

$$K^\varepsilon(x,t) = \int_{-\varepsilon}^{\varepsilon} \eta_\varepsilon(y) K(x-y,t)\, dy, \tag{8}$$

where $\eta_\varepsilon(x) = \frac{1}{\varepsilon}\eta(\frac{x}{\varepsilon})$ and $\eta(x)$ is any C^∞ even function with support on $[-1,1]$ that is monotonically decreasing on $[0,1]$ and has the property that $\int_{-\infty}^{\infty} \eta(x)\, dx = 1$. Note that $K^{\inf} \leqslant K^\varepsilon(x,t) \leqslant K^{\sup}$. Since we will only be interested in behavior as $\varepsilon \to 0$, we need only consider $\varepsilon \in (0, \varepsilon^{\max}]$, where $\varepsilon^{\max}$ is as small a positive number as desired. [As stated in the Introduction, while we use convolution with any standard mollifier in this paper, our proof can be easily extended to many other non-standard mollifiers (as, for example, were the two used in [18]). All that will really matter for our proof is that (1) K^ε is continuous, (2) K^ε converges to K uniformly on sets bounded away from the curves where K is discontinuous, and (3) at all (except possibly a finite number of) points of discontinuity (x_0, t_0), we have that

$$\lim_{\varepsilon\to 0}\left[\min_{x\in[x_0-\varepsilon, x_0+\varepsilon]} K^\varepsilon(x, t_0)\right] = \liminf_{x\to x_0} K(x, t_0)$$

and

$$\lim_{\varepsilon\to 0}\left[\max_{x\in[x_0-\varepsilon, x_0+\varepsilon]} K^\varepsilon(x, t_0)\right] = \limsup_{x\to x_0} K(x, t_0).$$

(So, essentially, near the discontinuity, K^ε does not stay below $\liminf K$ nor above $\limsup K$ as $\varepsilon \to 0$.)]

Since the K^ε are continuous, we can, for any ε, define $u^\varepsilon(x, t)$, the unique viscosity solution to

$$\begin{gathered} u_t^\varepsilon + H(K^\varepsilon(x, t), u_x^\varepsilon) = 0, \qquad x \in (-\infty, \infty), \qquad t \in [0, \infty), \\ u^\varepsilon(x, 0) = g(x). \end{gathered} \tag{9}$$

We will assume there is an a priori bound, $\hat{V} < \infty$, which is uniform in ε, on the absolute value of the characteristic speeds in these viscosity solutions.

We will be interested in the value of u, which we will define in the next section, and u^ε at an arbitrary but fixed point (X, T), so our analysis will take place on the region $\Omega \equiv \{(x, t)\colon\ x \in (-\infty, \infty),\ t \in [0, T]\}$. Also, it will be more convenient to use the functions

$$\begin{aligned} k(x, t) &\equiv K(x, T - t), \qquad 0 \leqslant t \leqslant T, \\ k^\varepsilon(x, t) &\equiv K^\varepsilon(x, T - t), \qquad 0 \leqslant t \leqslant T \end{aligned}$$

in place of $K(x, t)$ and $K^\varepsilon(x, t)$ throughout the remainder of this paper.

Next, we state our assumptions for the set $J \subset \Omega$ on which $k(x, t)$ is discontinuous. We will assume that on $\Omega - J$, $k(x, t)$ is Lipschitz continuous with Lipschitz constant l. We assume J is the union of a finite number, N, of curves, $\{\Gamma_i\}_{i=1}^N$. Each Γ_i is parameterized by t, i.e. for each i, there is some $t_i^{\min}$, $t_i^{\max}$, and function γ_i, where

$$\bar{\Gamma}_i = \{(\gamma_i(t), t)\colon\ t \in [t_i^{\min}, t_i^{\max}]\}. \tag{10}$$

(Note that (10) involves $\bar{\Gamma}_i$, the closure of Γ_i, so that each Γ_i is allowed to include or not include one or both of its endpoints, $(\gamma_i(t_i^{\min}), t_i^{\min})$ and

$(\gamma_i(t_i^{\max}), t_i^{\max})$.) We assume that each $\gamma_i \in C^2([t_i^{\min}, t_i^{\max}])$; therefore, there exists a bound on the derivative of γ_i, which we will denote by $||\gamma'||$ and a bound on the second derivative of γ_i, which we will denote by $||\gamma''||$. That is,

$$||\gamma'|| \equiv \max_{\substack{i=1,2,\dots,N \\ t\in[t_i^{\min}, t_i^{\max}]}} |\gamma_i'(t)|$$

and

$$||\gamma''|| \equiv \max_{\substack{i=1,2,\dots,N \\ t\in[t_i^{\min}, t_i^{\max}]}} |\gamma_i'(t)|.$$

We also assume that the Γ_i can only intersect at their endpoints. This formulation allows for a somewhat general set of discontinuities in k. Curves of discontinuity that fail to be C^2 at a finite number of locations or that intersect other discontinuity curves a finite number of times can generally be accommodated by being re-expressed as a union of other curves (e.g. see Fig. 1).

Now we review the control theory representation for $u^\varepsilon(X, T)$, the viscosity solution to (9). First, we define h, the cost function, by

$$h(k, -p) \equiv \max_{a\in[-W,W]} [ap - H(k, a)], \qquad p \in [-V, V], \tag{11}$$

where W is an arbitrarily large number and V is any number greater than both $\hat{V}$ and λ, the Lipschitz constant of $H(k, a)$ over the rectangular region $k \in [K^{\inf}, K^{\sup}]$, $a \in [-W, W]$. (We quickly note that h is very closely related

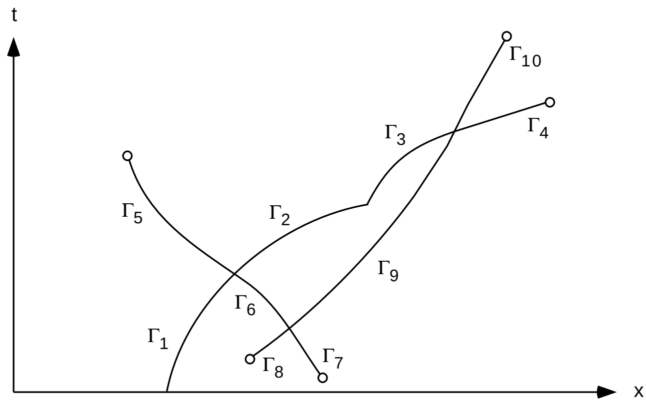

FIG. 1. Restricting curves of discontinuity, Γ_i, to be C^2 smooth and only intersect at endpoints has few practical implications since most curves can easily be re-expressed to conform to these restrictions. For example, the three curves of discontinuity shown appear to violate these restrictions at first, but they are perfectly allowable since they can be redefined as ten curves, $\Gamma_1, \Gamma_2, \dots, \Gamma_{10}$, which conform to our restrictions, as shown.

to the standard Lagrangian, L; specifically, $L(k,a) = h(k,-a)$.) By standard convex duality arguments we know that (1) $h(x,p)$ is convex in p, (2) $h(x,p)$ is Lipschitz continuous in both of its arguments (and we will use L to denote h's Lipschitz constant), and (3)

$$H(k,p) \equiv \max_{a\in[-V,V]} [ap - h(k,-a)], \qquad p \in [-W,W]. \tag{12}$$

The viscosity solution for (9) – where the $H(k,p)$ in (9) is replaced by the definition of $H(k,p)$ in (12) for *all* $p \in (-\infty,\infty)$ – is

$$u^\varepsilon(X,T) = \inf_{\alpha(\cdot)\in A} \left[\int_0^T h(k^\varepsilon(x(t),t),\alpha(t))\,dt + g(x(T)) \right], \tag{13}$$

where $x(t)$ is subject to the dynamics

$$\dot{x}(t) = \alpha(t), \qquad x(0) = X$$

and A is the set of all measurable functions from $[0,T]$ to $[-V,V]$. Since we want (13) to represent the actual solution to (9), we need to be able to find a value of W large enough so that altering $H(k,p)$ for $|p| > W$ by using (12) makes no difference; in other words we need to find a value of W so that $|u_x^\varepsilon(x,t)|$ – where u^ε is defined by either (9) or (13) – stays strictly below W. This means there must be a value of W large enough so that at no point in the rectangular domain $k \in [K^{\inf}, K^{\sup}]$, $p \in [-V,V]$ does $h(k,p)$ depend upon W. By assuming we have the superlinear growth condition

$$\lim_{p\to\pm\infty} \left| \frac{H(k,p)}{p} \right| = \infty \qquad \text{for any } k \in [K^{\inf}, K^{\sup}],$$

we guarantee from (11) that such a W exists, and therefore $u_x^\varepsilon(x,t)$ remains bounded. As stated in the previous section, if we know that such a W exists for other reasons, then this superlinear growth condition is no longer necessary.

We note that since $\alpha(t)$ corresponds to the characteristic speed of the solution (in fact $x(t)$ appears to be a backwards generalized characteristic curve emanating from (X,T) as defined by Dafermos [4]), by assumption $\alpha(t)$ will not take values outside of $[-\hat{V}, \hat{V}]$; however, we allow for hypothetical paths in the larger range $[-V,V]$ as it allows us to extend and simplify our proofs. Specifically, we will set

$$V \equiv 2\max\{\hat{V}, \lambda, \|\gamma'\|\}$$

for the remainder of this paper.

Finally, we define $\|h\|$ to be a bound on the absolute value of $h(\kappa,a)$ over the region $\kappa \in [K^{\inf}, K^{\sup}]$, $a \in [-V,V]$. This bound must exist since h is a continuous function. Note that since both $k(x,t)$ and $k^{\varepsilon}(x,t)$ are bounded by $K^{\inf}$ and $K^{\sup}$, $||h||$ bounds both $|h(k,\alpha)|$ and $|h(k^{\varepsilon},\alpha)|$.

3. DEFINITION AND PROPERTIES OF THE SOLUTION

We wish to show that as $\varepsilon \to 0$, $u^{\varepsilon}(X,T)$ converges pointwise to some function $u(X,T)$, which we can then define as the solution to (1). In fact, the solution $u(X,T)$ will have the following control theory representation:

$$u(X,T) = \inf_{\alpha(\cdot)\in A}\left[\int_0^T h(\hat{k}(x(t),t),\alpha(t))\,dt + g(x(T))\right],$$

where, as before, $x(t)$ is subject to the dynamics $\dot{x}(t)=\alpha(t)$, $x(0)=X$, A is the set of all measurable functions from $[0,T]$ to $[-V,V]$, and we now define $\hat{k}(x,t)$ for any $(x,t)\in\Omega$ by

$$\hat{k}(x,t) \equiv \begin{cases} k(x,t) & \text{if } (x,t)\in\Omega - J, \\ \operatorname{argmin}_{k\in[k_1(t),k_2(t)]}[h(k,\gamma'(t))] & \text{if } x=\gamma_i(t) \text{ and } t\in(t_i^{\min},t_i^{\max}), \\ \text{any value} \in [K^{\inf},K^{\sup}] & \text{if } x=\gamma_i(t) \text{ and } t=t_i^{\min} \text{ or } t_i^{\max}, \end{cases}$$

where $k_1(t)\equiv \liminf_{\substack{y\to\gamma_i(t)\\ y\neq\gamma_i(t)}} k(y,t)$ and $k_2(t)\equiv \limsup_{\substack{y\to\gamma_i(t)\\ y\neq\gamma_i(t)}} k(y,t)$. The first line of this definition just says that $\hat{k}=k$ at all points of continuity for k. The second line of this definition is the key line of this paper as it explicitly shows how information can travel along a curve of discontinuity. Specifically, at any point in a curve of discontinuity – excepting endpoints – we consider the two limits of k as we approach the point from either side of the curve of discontinuity. We let $\hat{k}$ equal the value of k that minimizes the cost function subject to being between (or equaling one of) these two limits. Finally, the third line of this definition assigns values to $\hat{k}$ at the endpoints of any curve of discontinuity (i.e. at $t_i^{\min}$ and $t_i^{\max}$). The actual values assigned to $\hat{k}$ at these endpoints are unimportant since the number of endpoints is finite, and therefore their value has no effect on the integrated cost. We only restrict the value of $\hat{k}$ at the endpoint to be between $K^{\inf}$ and $K^{\sup}$ because this allows some minor simplification in later arguments.

Our primary theorem, which we establish in the next section, is to show that for any $(X,T)\in\Omega$, we have that $\lim_{\varepsilon\to 0} u^{\varepsilon}(X,T) = u(X,T)$, thus establishing that the above expression for u is the unique stable solution. In Section 5 we will show that (1) $u^{\varepsilon}\to u$ uniformly on bounded subsets of Ω, and (2) u is Lipschitz continuous, which will follow from the primary theorem and establishing that the u^{ε} are equilipschitz.

4. PROOF OF THEOREM 1

In this section we establish the primary theorem of this paper:

THEOREM 1. $\lim_{\varepsilon \to 0} u^\varepsilon(X, T) = u(X, T)$.

4.1. *Sketch of the Proof*

We begin with an informal discussion of our proof. We first wish to establish that $\lim_{\varepsilon \to 0} u^\varepsilon(X, T) \geqslant u(X, T)$. To do this we look at (almost) cost minimizing paths, $x^\varepsilon(t)$ and $x(t)$, for $u^\varepsilon(X, T)$ and $u(X, T)$, respectively. We first consider the cost of $x^\varepsilon(t)$ as it progresses through three possible subregions of Ω and show that in each subregion a path $x(t)$ can be chosen, where any additional cost of the $x(t)$ path must shrink to zero as $\varepsilon \to 0$. When $x^\varepsilon(t)$ is in the first subregion of Ω (called Ω' in the next subsection) which stays away from the neighborhood of discontinuity curves, we can select $x(t)$ to follow the same path as $x^\varepsilon(t)$ at about the same cost (which becomes the same as $\varepsilon \to 0$). The second subregion (called J^δ in the next subsection) comprised of the δ neighborhoods of the $2N$ possible endpoints. Although the comparison of costs of $x^\varepsilon(t)$ and $x(t)$ is complex in this region due to the fact that discontinuity curves can intersect at endpoints, both the cost for $x^\varepsilon(t)$ and the cost for $x(t)$ become negligible as we consider progressively small values of δ.

This leads to the final subregion (called J^ε in the next subsection) which requires most of our analysis. This subregion is comprised of ε neighborhoods of the curves of discontinuities (minus the curves' endpoints, which were in J^δ). We chop each of these curves into tiny sections of length d. Since J^δ has been removed and each curve is smooth, d can be chosen so that each section is approximately linear. This allows us to essentially reduce the problem to the case of comparing the costs of $x^\varepsilon(t)$ and $x(t)$ near a straight line of discontinuity with constant values k_1 and k_2 on either side of the discontinuity. This construction is at the heart of the proof because Jensen's inequality can be used (as the cost function $h(k, \alpha)$ is convex in α) to show that if $x^\varepsilon(t)$ stays near the discontinuity, it reduces costs by running parallel to the line of discontinuity along the line where the value of $k^\varepsilon = \arg\min_{k^\varepsilon \in [k_1, k_2]} [h(k^\varepsilon, \alpha_0)]$ where α_0 corresponds to the direction of the line of discontinuity. Since $x(t)$ can follow almost this same path at the same cost, we will see that $\liminf_{\varepsilon \to 0} u^\varepsilon(X, T) \geqslant u(X, T)$ by first letting $\varepsilon \to 0$, then letting $d \to 0$, and finally letting $\delta \to 0$.

By a similar argument which will analyze $x(t)$ and show that $x^\varepsilon(t)$ can be chosen so that its additional cost shrinks to zero as $\varepsilon \to 0$, we will establish $\limsup_{\varepsilon \to 0} u^\varepsilon(X, T) \leqslant u(X, T)$ completing the proof.

4.2. Definitions

The following definitions will be useful in our proof:

(1) We now explicitly partition Ω into the following three regions:

$$\Omega = J^{\varepsilon} \cup J^{\delta} \cup \Omega',$$

where $J^{\varepsilon}, J^{\delta}$, and Ω' are defined by

- $J^{\varepsilon} \equiv \bigcup_{i=1}^{N} \Gamma_i^{\delta,\varepsilon}$ where

$$\Gamma_i^{\delta,\varepsilon} \equiv \{(x+\xi, t)\colon\ (x,t) \in \Gamma_i^{\delta} \ \text{ and } \ \xi \in [-\varepsilon, \varepsilon]\}$$

and $\Gamma_i^{\delta} \equiv \{(\gamma_i(t), t)\colon\ t \in [t_i^{\min} + \delta, t_i^{\max} - \delta]\}$. In other words, J^{ε} is the ε neighborhood of J after we remove all regions within δ of any of the discontinuity curves' endpoints.

- $J^{\delta} \equiv \bigcup_{i=1}^{N} \left\{ \begin{smallmatrix} (\gamma_i(t)+\xi, t)\colon\ \xi \in [-\varepsilon,\varepsilon] \ \text{and} \\ t \in [t_i^{\min}, t_i^{\min}+\delta) \cup (t_i^{\max}-\delta, t_i^{\max}] \end{smallmatrix} \right\}$. In other words, J^{δ} is the ε neighborhood of the points of discontinuity within δ of any of the discontinuity curves' endpoints.
- $\Omega' \equiv \Omega - (J^{\varepsilon} \cup J^{\delta})$.

By choosing $\varepsilon^{\max}$ sufficiently small, we are guaranteed that J^{ε}, J^{δ}, and Ω' do not intersect each other.

(2) Also we define Γ^d to represent any generic subsection of "length" d of any Γ_i^{δ}. Specifically, for some $i \in \{1, 2, \ldots, N\}$ and t_0 where $[t_0, t_0 + d] \subset [t_i^{\min} + \delta, t_i^{\max} - \delta]$

$$\Gamma^d \equiv \{(\gamma_i(t), t)\colon\ t \in [t_0, t_0 + d]\}.$$

The definition of the region $\Gamma^{d,\varepsilon}$ is analogous to the definition of the region $\Gamma_i^{\delta,\varepsilon}$:

$$\Gamma^{d,\varepsilon} \equiv \{(x+\xi, t)\colon\ (x,t) \in \Gamma^d \qquad \text{and} \qquad \xi \in [-\varepsilon, \varepsilon]\}.$$

We will eventually shrink d to zero before shrinking δ to zero; therefore, we can allow the maximum size of d to be restricted by the value of δ at intermediate steps.

4.3. Proof that $\liminf_{\varepsilon \to 0} u^{\varepsilon}(X,T) \geqslant u(X,T)$

We now define $x^{\varepsilon}(t)$, a family of paths parameterized by ε, such that

$$u^{\varepsilon}(X,T) \geqslant \int_0^T h(k^{\varepsilon}(x^{\varepsilon}(t), t), \alpha^{\varepsilon}(t))\, dt + g(x^{\varepsilon}(T)) - \varepsilon, \tag{14}$$

where $\dot{x}^{\varepsilon}(t) = \alpha^{\varepsilon}(t)$, $x^{\varepsilon}(0) = X$, and $|\alpha^{\varepsilon}(t)| \leqslant V$. Since we will eventually shrink ε to zero before shrinking d and δ to zero, we can choose $\varepsilon^{\max}$ to be arbitrarily small and depend on the values of d and δ. In our analysis, however, we will generally think of ε as fixed (though small), and we will be interested in comparing $\int_0^T h(k^\varepsilon(x^\varepsilon(t),t),\alpha^\varepsilon(t))\,dt$, the cost corresponding to $x^\varepsilon(t)$, to $\int_0^T h(\hat{k}(x(t),t),\alpha(t))\,dt$, the cost corresponding to $x(t)$, which is a path we will construct subject to the constraints $\dot{x}(t)=\alpha(t)$, $x(0)=X$, $x(T)=x^\varepsilon(T)$, and $|\alpha(t)|\leqslant V$. Before constructing $x(t)$, however, we will first consider the possible cost of $x^\varepsilon(t)$ when it intersects J^ε.

We begin by restricting $\varepsilon^{\max}$ so that $\varepsilon^{\max}<\frac{Vd}{8}$ and restricting d so that $d<\frac{D(\delta)}{V}$, where $D(\delta)$ is the following measure of the closest the x-distance between any Γ_i^δ and Γ_j can be:

$$D(\delta)\equiv\min_{i\neq j}\{|\gamma_i(t)-\gamma_j(t)|: t\in[t_i^{\min}+\delta,t_i^{\max}-\delta]\cap[t_j^{\min},t_j^{\max}]\}.$$

(Note that since it is assumed that none of the Γ_i can intersect each other except possibly at endpoints, it must be the case that $D(\delta)>0$ if $\delta>0$.) These restrictions on $\varepsilon^{\max}$ and d isolate each of the $\Gamma^{d,\varepsilon}$. Specifically, if at $t=t_0$ we have a path ($x^\varepsilon(t)$ or $x(t)$) intersecting $\Gamma_i^{\delta,\varepsilon}$, then the path cannot come within an x-distance of ε from any Γ_j where $i\neq j$ and then return to $\Gamma_i^{\delta,\varepsilon}$ when $t\leqslant t_0+d$. Therefore, all intersections of $x^\varepsilon(t)$ with J^ε are with a single $\Gamma^{d,\varepsilon}$ when $t\in[t^{in},t^{out}]$ where t^{in} is defined to be any time where $x^\varepsilon(t)$ intersects some $\Gamma_i^{\delta,\varepsilon}$ (so $(x^\varepsilon(t^{in}),t^{in})\in\Gamma_i^{\delta,\varepsilon}$) and t^{out} is defined to be the last time that $x^\varepsilon(t)$ intersects $\Gamma_i^{\delta,\varepsilon}$ subject to $t\leqslant t^{in}+d$ (i.e. $t^{out}\equiv\max_{t\in[t^{in},t^{in}+d]}\{t\colon (x^\varepsilon(t),t)\in\Gamma_i^{\delta,\varepsilon}\}$).

We are now ready to analyze $\int_{t^{in}}^{t^{out}} h(k^\varepsilon(x^\varepsilon(t),t),\alpha^\varepsilon(t))\,dt$, the cost of $x^\varepsilon(t)$ between t^{in} and t^{out}. First, we define two functions, $\hat{k}'(x,t)$ and $k^{\varepsilon,n}(x,t)$, over the region $\Gamma^{d,\frac{3Vd}{4}+2\varepsilon}\equiv\{(\gamma(t)+x,t)\colon t\in[t^{in},t^{out}]$ and $x\in[-(\frac{3Vd}{4}+2\varepsilon),(\frac{3Vd}{4}+2\varepsilon)]\}$, which will take values similar to $\hat{k}$ and k^ε, respectively, but are much easier to analyze. (The region is chosen because (1) $\Gamma^{d,\frac{3Vd}{4}+2\varepsilon}$ cannot intersect any Γ_j where $i\neq j$, and (2) $(x^\varepsilon(t),t)\in\Gamma^{d,\frac{3Vd}{4}+\varepsilon}$ for all $t\in[t^{in},t^{out}]$, where $\Gamma^{d,\frac{3Vd}{4}+\varepsilon}$ has the same definition as $\Gamma^{d,\frac{3Vd}{4}+2\varepsilon}$ but with 2ε replaced by ε.) We define the function $\hat{k}$, which is a piecewise constant approximation to $\hat{k}$, by

$$\hat{k}'(x,t)\equiv\begin{cases}\lim_{x\to\gamma(t^{in})^-}k(x,t^{in}) & \text{if } x<\gamma(t),\\ \lim_{x\to\gamma(t^{in})^+}k(x,t^{in}) & \text{if } x>\gamma(t),\\ \hat{k}(x,t) & \text{if } x=\gamma(t).\end{cases}$$

Note that for any $(x,t)\in\Gamma^{d,\frac{3Vd}{4}+2\varepsilon}$ we have that

$$\left|\hat{k}'(x,t)-\hat{k}(x,t)\right|\leqslant ld\sqrt{\frac{9V^2}{4}+1} \tag{15}$$

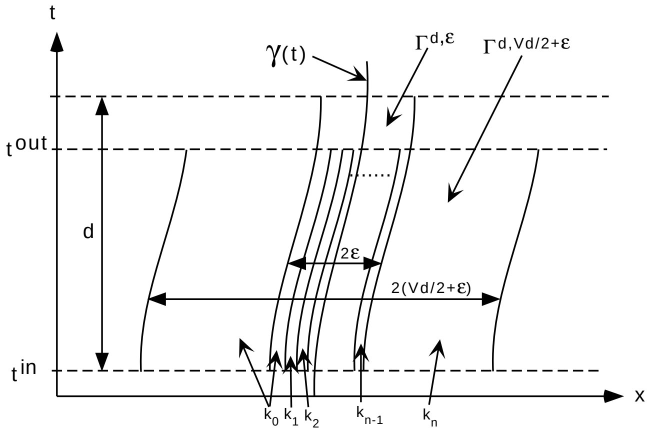

FIG. 2. Structures near $\Gamma^{d,\varepsilon}$: $\Gamma^{d,\varepsilon}$ is the 2ε thick band about $\gamma(t)$. Between t^{in} and t^{out}, $x^{\varepsilon}(t)$ must stay within $\Gamma^{d,\frac{Vd}{2}+\varepsilon}$, the $2(\frac{Vd}{2}+\varepsilon)$ thick band about $\gamma(t)$. The function $k^{\varepsilon,n}$ takes values $k_1, k_2, \ldots, k_{n-1}$ along the bands within $\Gamma^{d,\varepsilon}$ between t^{in} and t^{out}. These values monotonically increase or decrease between k_0 and k_n, which roughly correspond to the limit of $\hat{k}$ as $x \to \gamma(t)$ from the left and right, respectively.

from the assumption that k (and therefore $\hat{k}$) is Lipschitz continuous on $\Omega - J$. Now we can define $k^{\varepsilon,n}$:

$$k^{\varepsilon,n}(x,t) \equiv \begin{cases} \hat{k}'(x,t) & \text{if } |x-\gamma(t)| \geqslant \varepsilon, \\ \int_{-\varepsilon}^{\varepsilon} \hat{k}'(\langle x\rangle - y, t)\eta_\varepsilon(y)\,dy & \text{if } |x-\gamma(t)| < \varepsilon, \end{cases}$$

where $\langle x\rangle \equiv \max_{i\in\{0,1,\ldots\}}\{\gamma(t)-\varepsilon+i\frac{2\varepsilon}{n} : \gamma(t)-\varepsilon+i\frac{2\varepsilon}{n}<x\}$. That is, $k^{\varepsilon,n}$ is a piecewise constant function that takes the value $k_0 \equiv \lim_{x\to\gamma(t^{in})^-} k(x,t^{in})$ to the "left" of $\Gamma^{d,\varepsilon}$, the value $k_n \equiv \lim_{x\to\gamma(t^{in})^+} k(x,t^{in})$ to the "right" of $\Gamma^{d,\varepsilon}$, and values $k_1, k_2, \ldots k_{n-1}$ within $\Gamma^{d,\varepsilon}$ which form a monotonic "staircase" between the values k_0 and k_n (so, to be precise,

$$k_i \equiv \int_{-\varepsilon}^{\varepsilon} \hat{k}'\left(\gamma(t)-\varepsilon+i\frac{2\varepsilon}{n}-y, t\right)\eta_\varepsilon(y)\,dy, \qquad i=1,2,\ldots,n-1).$$

Fig. 2 gives a geometric representation of $k^{\varepsilon,n}$.

Note that for any $(x,t) \in \Gamma^{d,\frac{3Vd}{4}+\varepsilon}$

$$\lim_{n\to\infty} |k^{\varepsilon,n}(x,t) - k^{\varepsilon}(x,t)| \leqslant ld\sqrt{\frac{9V^2}{4}+1}. \tag{16}$$

This follows from the fact that $\lim_{n\to\infty} k^{\varepsilon,n}(x,t) = \int_{-\varepsilon}^{\varepsilon} \hat{k}'(x-y,t)\eta_\varepsilon(y)\,dy$ and therefore,

$$\begin{aligned}
&\lim_{n\to\infty} |k^{\varepsilon,n}(x,t) - k^\varepsilon(x,t)| \\
&\quad = \left| \int_{-\varepsilon}^{\varepsilon} \hat{k}'(x-y,t)\eta_\varepsilon(y)\,dy - \int_{-\varepsilon}^{\varepsilon} \hat{k}(x-y,t)\eta_\varepsilon(y)\,dy \right| \\
&\quad \leqslant \int_{-\varepsilon}^{\varepsilon} |\hat{k}'(x-y,t) - \hat{k}(x-y,t)|\eta_\varepsilon(y)\,dy \\
&\quad \leqslant ld\sqrt{\frac{9V^2}{4}+1},
\end{aligned}$$

where the last inequality is justified by (15) and the fact that $\int_{-\varepsilon}^{\varepsilon} \eta_\varepsilon(y)\,dy = 1$.

Now we analyze the cost of $x^\varepsilon(t)$ on $k^{\varepsilon,n}(x,t)$, which we will later connect to the cost of $x^\varepsilon(t)$ on $k^\varepsilon(x,t)$ using (16). Since $(x^\varepsilon(t),t) \in \Gamma^{d,\frac{3Vd}{4}+\varepsilon}$ for all $t \in [t^{in}, t^{out}]$, we can define the following disjoint partition of $[t^{in}, t^{out}]$:

$$Q_i = \{t\colon\ k^{\varepsilon,n}(x^\varepsilon(t),t) = k_i\}, \qquad i = 0,1,\ldots,n$$

and since $[t^{in}, t^{out}] = \bigcup_{i=0}^{n} Q_i$, we have that

$$\sum_{i=0}^{n} m(Q_i) = t^{out} - t^{in},$$

where $m(Q_i)$ is the outer measure of Q_i.

Creating this partition allows us to apply Jensen's inequality, which is the key step in our analysis:

$$\begin{aligned}
\int_{t^{in}}^{t^{out}} h(k^{\varepsilon,n}(x^\varepsilon(t),t),\alpha^\varepsilon(t))\,dt &= \sum_{i=0}^{n} \int_{Q_i} h(k_i,\alpha^\varepsilon(t))\,dt \\
&\geqslant \sum_{i=0}^{n} h\left(k_i, \frac{\int_{Q_i}\alpha^\varepsilon(t)\,dt}{m(Q_i)}\right) m(Q_i). \qquad (17)
\end{aligned}$$

This also motivates our next step which is to analyze $\int_{Q_i} \alpha^\varepsilon(t)\,dt$ for $i = 0,1,\ldots,n$.

We begin with $\int_{Q_n} \alpha^\varepsilon(t)\,dt$. The boundary of the region within $\Gamma^{d,\frac{3Vd}{4}+2\varepsilon}$ where $k^{\varepsilon,n}(x,t) = k_n$ is parallel to $\Gamma^d = (\gamma(t),t)$, the discontinuity (where $\gamma(t)$ denotes the $\gamma_i(t)$ of interest). Since $x^\varepsilon(t^{in})$ and $x^\varepsilon(t^{out}) \in \Gamma^{d,\varepsilon}$, t^{in} (or t^{out}) can only be in Q_n if $(x^\varepsilon(t^{in}),t^{in})$ (or $(x^\varepsilon(t^{out}),t^{out})$) are on this boundary, and so we have that the path $x^\varepsilon(t)$ must be on this boundary at any $t \in Q_n$ that is not in the interior of Q_n (typically, endpoints of intervals in Q_n). Therefore, since $\alpha^\varepsilon(t) = \dot{x}^\varepsilon(t)$, we must have that

$$\int_{Q_n} \alpha^\varepsilon(t)\, dt = \int_{Q_n} \gamma'(t)\, dt. \tag{18}$$

The analysis of $\int_{Q_0} \alpha^\varepsilon(t)\, dt$ is similar. However, unlike the Q_n case, the region where $k^{\varepsilon,n}(x,t) = k_0$ overlaps $\Gamma^{d,\varepsilon}$ is a strip running parallel to Γ^d with a constant width in the x direction of $\frac{2\varepsilon}{n}$. Therefore, $(x^\varepsilon(t^{in}), t^{in})$ or $(x^\varepsilon(t^{out}), t^{out})$ may be within this strip and so

$$\int_{Q_0} \alpha^\varepsilon(t)\, dt = \int_{Q_0} \gamma'(t)\, dt + \Delta_0, \tag{19}$$

where $|\Delta_0| \leqslant \frac{2\varepsilon}{n}$.

For $\int_{Q_i} \alpha^\varepsilon(t)\, dt$ where $i = 1, 2, \ldots, n-1$, we have that the region where $k^{\varepsilon,n}(x,t) = k_i$ is a strip running parallel to Γ^d with a $\frac{2\varepsilon}{n}$ width, and so there are two boundaries parallel to Γ^d that can be crossed. This poses no additional difficulties; however, because any x distance gained by crossing from the left boundary to the right boundary must be lost (by going from the right boundary to the left boundary) before it can be gained again, and so we still have

$$\int_{Q_i} \alpha^\varepsilon(t)\, dt = \int_{Q_i} \gamma'(t)\, dt + \Delta_i, \qquad i = 1, 2, \ldots, n-1, \tag{20}$$

where $|\Delta_i| \leqslant \frac{2\varepsilon}{n}$.

Because $|\gamma''(t)|$ is bounded by $\|\gamma''\|$, we have for any $t \in [t^{in}, t^{out}]$

$$|\gamma'(t) - \gamma'(t^{in})| \leqslant \int_{t^{in}}^{t} |\gamma''(t)| dt \leqslant \|\gamma''\| d \tag{21}$$

since $t^{out} - t^{in} \leqslant d$. Subtracting $\gamma'(t^{in}) m(Q_i)$ from (18)–(20) and then applying (21) yields

$$\left| \int_{Q_i} \alpha^\varepsilon(t)\, dt - \gamma'(t^{in}) m(Q_i) \right| \leqslant \|\gamma''\| d m(Q_i) + |\Delta_i|, \qquad i = 0, 1, \ldots, n, \tag{22}$$

where $\Delta_n = 0$.

Now, recalling that L is the Lipschitz constant of h, we are ready to return to our goal of determining the cost of $x^\varepsilon(t)$ on $k^\varepsilon(x,t)$ between t^{in} and t^{out}:

$$
\begin{aligned}
&\int_{t^{in}}^{t^{out}} h(k^{\varepsilon}(x^{\varepsilon}(t),t),\alpha^{\varepsilon}(t))\,dt \\
&\quad = \int_{t^{in}}^{t^{out}} h(k^{\varepsilon,n}(x^{\varepsilon}(t),t) - k^{\varepsilon,n}(x^{\varepsilon}(t),t) - k^{\varepsilon}(x^{\varepsilon}(t),t),\alpha^{\varepsilon}(t))\,dt \\
&\quad \geqslant \int_{t^{in}}^{t^{out}} h(k^{\varepsilon,n}(x^{\varepsilon}(t),t),\alpha^{\varepsilon}(t))\,dt - L\int_{t^{in}}^{t^{out}} |k^{\varepsilon,n}(x^{\varepsilon}(t),t) - k^{\varepsilon}(x^{\varepsilon}(t),t)|\,dt. \qquad (23)
\end{aligned}
$$

Using (17) and (22) we can further analyze the first integral in the last line of (23):

$$
\begin{aligned}
&\int_{t^{in}}^{t^{out}} h(k^{\varepsilon,n}(x^{\varepsilon}(t),t),\alpha^{\varepsilon}(t))\,dt \\
&\quad \geqslant \sum_{i=0}^{n} h\left(k_i, \frac{\int_{Q_i}\alpha^{\varepsilon}(t)\,dt}{m(Q_i)}\right)m(Q_i) \\
&\quad = \sum_{i=0}^{n} h\left(k_i, \frac{\gamma'(t^{in})m(Q_i) - [\gamma'(t^{in})m(Q_i) - \int_{Q_i}\alpha^{\varepsilon}(t)\,dt]}{m(Q_i)}\right)m(Q_i) \\
&\quad \geqslant \sum_{i=0}^{n}\left[h(k_i,\gamma'(t^{in}))m(Q_i) - L\left|\int_{Q_i}\alpha^{\varepsilon}(t)\,dt - \gamma'(t^{in})m(Q_i)\right|\right] \\
&\quad \geqslant \left[\sum_{i=0}^{n} h(k_i,\gamma'(t^{in}))m(Q_i)\right] - \|\gamma''\|Ld^2 - 2L\varepsilon. \qquad (24)
\end{aligned}
$$

Defining $k(t^{in}) \equiv \arg\min_{k\in[k_0,k_n]}[h(k,\gamma'(t^{in}))]$, we have that

$$h(k_i,\gamma'(t^{in})) \geqslant h(k(t^{in}),\gamma'(t^{in}))$$

and therefore we see that for the summation in the last line of (24)

$$\sum_{i=0}^{n} h(k_i,\gamma'(t^{in}))m(Q_i) \geqslant h(k(t^{in}),\gamma'(t^{in}))(t^{out} - t^{in}).$$

Substituting this in (24) and then substituting (24) into (23) yields

$$
\begin{aligned}
&\int_{t^{in}}^{t^{out}} h(k^{\varepsilon}(x^{\varepsilon}(t),t),\alpha^{\varepsilon}(t))\,dt \\
&\quad \geqslant h(k(t^{in}),\gamma'(t^{in}))(t^{out} - t^{in}) - \|\gamma''\|Ld^2 - 2L\varepsilon \\
&\qquad - L\int_{t^{in}}^{t^{out}} |k^{\varepsilon,n}(x^{\varepsilon}(t),t) - k^{\varepsilon}(x^{\varepsilon}(t),t)|\,dt. \qquad (25)
\end{aligned}
$$

Next, we take the limit as $n \to \infty$ of (25) and apply first the Lebesgue dominated convergence theorem and then (16) to obtain

$$\int_{t^{in}}^{t^{out}} h(k^{\varepsilon}(x^{\varepsilon}(t), t), \alpha^{\varepsilon}(t))\, dt \geqslant h(k(t^{in}), \gamma'(t^{in}))(t^{out} - t^{in}) - Cd^2 - 2L\varepsilon, \quad (26)$$

where $C = \|\gamma''\|L + Ll\sqrt{\frac{9V^2}{4} + 1}$. Note that C is independent of d, ε, and δ.

Now, we construct a "cheap" path $x(t)$ for comparison to the cost of the $x^{\varepsilon}(t)$ path in (26). We let $x(t)$ and $x^{\varepsilon}(t)$ start and stop at the same location, therefore $x(t^{in}) = x^{\varepsilon}(t^{in})$ and $x(t^{out}) = x^{\varepsilon}(t^{out})$. We choose $x(t)$ to move as quickly as it can – that is, at speed V – from $(x^{\varepsilon}(t^{in}), t^{in})$ to the discontinuity, so this will take, at most, $\frac{2\varepsilon}{V}$ time. We then choose $x(t)$ to stay on the curve of discontinuity as long as possible before moving as quickly as it can from the curve of discontinuity to the point $(x^{\varepsilon}(t^{out}), t^{out})$ (which will again take, at most, $\frac{2\varepsilon}{V}$ time). The cost of this is bounded by

$$\begin{aligned}\int_{t^{in}}^{t^{out}} h(\hat{k}(x(t), t), \alpha(t))\, dt &\leqslant \frac{4\varepsilon}{V}\|h\| + \int_{t^{in}+\frac{2\varepsilon}{V}}^{t^{out}-\frac{2\varepsilon}{V}} h(\hat{k}(\gamma(t), t), \gamma'(t))\, dt \\ &\leqslant \frac{8\varepsilon}{V}\|h\| + \int_{t^{in}}^{t^{out}} h(\hat{k}(\gamma(t), t), \gamma'(t))\, dt. \end{aligned} \quad (27)$$

Now we compare $h(\hat{k}(\gamma(t), t), \gamma'(t))$ to $h(k(t^{in}), \gamma'(t^{in}))$. Recalling that for any $t \in [t^{in}, t^{out}]$, $\hat{k}(\gamma(t), t) = \arg\min_{k \in [k_1(t), k_2(t)]}[h(k, \gamma'(t))]$ where $k_1(t) \equiv \liminf_{\substack{y \to \gamma_i(t) \\ y \neq \gamma_i(t)}} k(y, t)$ and $k_2(t) \equiv \limsup_{\substack{y \to \gamma_i(t) \\ y \neq \gamma_i(t)}} k(y, t)$. and also noting that $k(t^{in}) = \hat{k}(\gamma(t^{in}), t^{in})$, we have that for any $t \in [t^{in}, t^{out}]$

$$\begin{aligned} h(\hat{k}(\gamma(t), t), \gamma'(t)) &= h\left(\arg\min_{k \in [k_1(t), k_2(t)]}[h(k, \gamma'(t))], \gamma'(t)\right) \\ &\leqslant h\left(\arg\min_{k \in [k_1(t), k_2(t)]}[h(k, \gamma'(t^{in}))], \gamma'(t)\right) \\ &\leqslant h\left(\arg\min_{k \in [k_1(t), k_2(t)]}[h(k, \gamma'(t^{in}))], \gamma'(t^{in})\right) + L\|\gamma''\|d \\ &\leqslant h\left(\arg\min_{k \in [k_1(t^{in}), k_2(t^{in})]}[h(k, \gamma'(t^{in}))], \gamma'(t^{in})\right) \end{aligned}$$

$$+\ L\|\gamma''\|d + Ll\sqrt{\frac{V^2}{4}+1}d$$

$$= h(k(t^{in}), \gamma'(t^{in})) + L\left(\|\gamma''\| + l\sqrt{\frac{V^2}{4}+1}\right)d, \tag{28}$$

where the middle inequality follows from (21) and the last inequality follows from k being Lipschitz on $\Omega - J$.

Combining (27) and (28) yields our cost result for $x(t)$:

$$\int_{t^{in}}^{t^{out}} h(\hat{k}(x(t),t),\alpha(t))\,dt \leqslant h(k(t^{in}),\gamma'(t^{in}))(t^{out}-t^{in})$$
$$+\ L\left(\|\gamma''\| + l\sqrt{\frac{V^2}{4}+1}\right)d^2 + \frac{8\|h\|}{V}\varepsilon \tag{29}$$

and so we can subtract the costs of the $x^{\varepsilon}(t)$ and $x(t)$ paths in (26) and (29) to obtain

$$\int_{t^{in}}^{t^{out}} h(k^{\varepsilon}(x^{\varepsilon}(t),t),\alpha^{\varepsilon}(t)) - h(\hat{k}(x(t),t),\alpha(t))\,dt \geqslant -C_1 d^2 - C_2\varepsilon, \tag{30}$$

where $C_1 \equiv C + L(\|\gamma''\| + l\sqrt{\frac{V^2}{4}+1})$ and $C_2 \equiv 2L + \frac{8\|h\|}{V}$. Note that both C_1 and C_2 are independent of d, ε, and δ.

Next, we divide $[0,T]$ into three categories: $\tau^{\varepsilon}, \tau^{\delta}$, and τ'. We define τ^{ε} to be the union of $[t^{in}, t^{out}]$ from each of the different $\Gamma^{d,\varepsilon}$ strips. Note that there are, at most, $2(\frac{T}{d}+N)$ different $\Gamma^{d,\varepsilon}$ (the "2" is necessary to accommodate paths that "ping-pong" between different $\Gamma_i^{\delta,\varepsilon}$); therefore, from (30), we have

$$\int_{\tau^{\varepsilon}} h(k^{\varepsilon}(x^{\varepsilon}(t),t),\alpha^{\varepsilon}(t)) - h(\hat{k}(x(t),t),\alpha(t))\,dt$$
$$\geqslant -2C_1(Td + Nd^2) - 2C_2\left(\frac{T}{d}+N\right)\varepsilon. \tag{31}$$

For all $t \in [0,T] - \tau^{\varepsilon}$, we choose the path $x(t)$ so that $x(t) = x^{\varepsilon}(t)$. We now define τ^{δ} to be the set of times when $x(t)$ (and therefore $x^{\varepsilon}(t)$ as well) are in J^{δ}. (i.e. $\tau^{\delta} \equiv \{t\colon (x(t),t) \in J^{\delta}\}$.) Since $m(\tau^{\delta}) \leqslant 2N\delta$, we have that

$$\int_{\tau^{\delta}} h(k^{\varepsilon}(x^{\varepsilon}(t),t),\alpha^{\varepsilon}(t)) - h(\hat{k}(x(t),t),\alpha(t))\,dt \geqslant -4\|h\|N\delta. \tag{32}$$

Finally, we define $\tau' = [0,T] - \tau^{\varepsilon} - \tau^{\delta}$. Note that if $t \in \tau'$, then $(x(t),t)$ and $(x^{\varepsilon}(t),t)$ must be in Ω'. Since Ω' stays away from the ε neighborhood of J, we have that, for any $(x,t) \in \Omega'$

$$\begin{aligned}
|k^\varepsilon(x,t) - \hat{k}(x,t)| &= \left| \int_{-\varepsilon}^{\varepsilon} k(x-y,t)\eta_\varepsilon(y) - k(x,t)\eta_\varepsilon(y)\, dy \right| \\
&\leqslant \int_{-\varepsilon}^{\varepsilon} |k(x-y,t) - k(x,t)|\eta_\varepsilon(y)\, dy \\
&\leqslant \int_{-\varepsilon}^{\varepsilon} l\varepsilon\eta_\varepsilon(y)\, dy \\
&= l\varepsilon. \qquad (33)
\end{aligned}$$

Therefore, from (33), the cost over τ' is bounded by

$$\begin{aligned}
&\int_{\tau'} h(k^\varepsilon(x^\varepsilon(t),t), \alpha^\varepsilon(t)) - h(\hat{k}(x(t),t), \alpha(t))\, dt \\
&\quad = \int_{\tau'} h(k^\varepsilon(x(t),t), \alpha(t)) - h(\hat{k}(x(t),t), \alpha(t))\, dt \\
&\quad \geqslant -LTl\varepsilon. \qquad (34)
\end{aligned}$$

Since $[0,T] = \tau^\varepsilon \cup \tau^\delta \cup \tau'$, we combine (31), (32), and (34) to obtain

$$\begin{aligned}
&\int_0^T h(k^\varepsilon(x^\varepsilon(t),t), \alpha^\varepsilon(t)) - h(\hat{k}(x(t),t), \alpha(t))\, dt \\
&\quad \geqslant -2C_1(Td + Nd^2) - \left(2C_2\left(\frac{T}{d} + N\right) + LTl\right)\varepsilon - 4\|h\|N\delta. \qquad (35)
\end{aligned}$$

Since it must be the case that

$$u(X,T) \leqslant \int_0^T h(\hat{k}(x(t),t), \alpha(t))\, dt + g(x(T)) \qquad (36)$$

and we have chosen $x(t)$ so that $x(T) = x^\varepsilon(T)$, we have from (14), (35), and (36) that

$$\begin{aligned}
u^\varepsilon(X,T) - u(X,T) \geqslant &-2C_1(Td + Nd^2) \\
&- \left(2C_2\left(\frac{T}{d} + N\right) + LTl + 1\right)\varepsilon - 4\|h\|N\delta.
\end{aligned}$$

Now we let ε shrink to zero (along a subsequence if necessary) yielding

$$\liminf_{\varepsilon \to 0} u^\varepsilon(X,T) - u(X,T) \geqslant -2C_1(Td + Nd^2) - 4\|h\|N\delta.$$

With ε infinitesimally small, we can consider d as small as we wish, so

$$\liminf_{\varepsilon\to 0} u^{\varepsilon}(X,T) - u(X,T) \geqslant -4\|h\|N\delta$$

and with d infinitesimally small, we can consider δ as small as we wish, which yields our desired result:

$$\liminf_{\varepsilon\to 0} u^{\varepsilon}(X,T) \geqslant u(X,T). \tag{37}$$

4.4. *Proof that* $\limsup_{\varepsilon\to 0} u^{\varepsilon}(X,T) \leqslant u(X,T)$

The argument here will heavily parallel the argument in the previous subsection where we established that $\liminf_{\varepsilon\to 0} u^{\varepsilon}(X,T) \geqslant u(X,T)$.

We begin by defining a sequence of paths, $x^n(t)$, where $n = 1, 2, \ldots$, such that

$$u(X,T) \geqslant \int_0^T h(\hat{k}(x^n(t),t), \alpha^n(t))\,dt + g(x^n(T)) - \frac{1}{n}, \tag{38}$$

where $\dot{x}^n(t) = \alpha^n(t)$, $x^n(0) = X$, and $|\alpha^n(t)| \leqslant V$. Our analysis of the $x^n(t)$ parallels the analysis of the $x^{\varepsilon}(t)$ in the previous subsection.

We begin, as before, by analyzing $x^n(t)$ in a $\Gamma^{d,\frac{3Vd}{4}+\varepsilon}$ region. We divide $[t^{in}, t^{out}] = Q^1 \cup Q^2 \cup Q^3$, where each $Q^i \subset [t^{in}, t^{out}]$ and

$$Q^1 \equiv \{t\colon x^n(t) < \gamma(t)\},$$

$$Q^2 \equiv \{t\colon x^n(t) = \gamma(t)\},$$

$$Q^3 \equiv \{t\colon x^n(t) > \gamma(t)\}.$$

Also, we define $k^1 \equiv \lim_{x\to\gamma(t^{in})^-} \hat{k}(x, t^{in}), k^2 \equiv \hat{k}(\gamma(t^{in}), t^{in})$, and $k^3 \equiv \lim_{x\to\gamma(t^{in})^+} \hat{k}(x, t^{in})$. From (15) we have that for any $t \in Q^1$

$$\left|k^1 - \hat{k}(x^n(t),t)\right| \leqslant ld\sqrt{\frac{9V^2}{4} + 1} \tag{39}$$

and for any $t \in Q^3$

$$\left|k^3 - \hat{k}(x^n(t),t)\right| \leqslant ld\sqrt{\frac{9V^2}{4} + 1}. \tag{40}$$

Repeating the same argument used in (17)–(24) to Q^1 and Q^3, we see that, parallel to (23),

$$\int_{Q^i} h(\hat{k}(x^n(t), t), \alpha^n(t))\, dt \geqslant \int_{Q^i} h(k^i, \alpha^n(t))\, dt - L\int_{Q^i} |k^i - \hat{k}(x^n(t), t)|\, dt, \qquad i = 1 \text{ or } 3 \tag{41}$$

and, parallel to (24),

$$\int_{Q^i} h(k^i, \alpha^n(t))\, dt \geqslant h(k^i, \gamma'(t^{in}))m(Q_i) - \|\gamma''\| Ldm(Q^i) - L\varepsilon, \qquad i = 1 \text{ or } 3. \tag{42}$$

Now we look more carefully at Q^2. The set of $t \in Q^2$ where $\alpha^n(t) \neq \gamma'(t)$ must have measure zero, therefore

$$\int_{Q^2} h(\hat{k}(x^n(t), t), \alpha^n(t))\, dt = \int_{Q^2} h(\hat{k}(x^n(t), t), \gamma'(t))\, dt. \tag{43}$$

Now we slightly alter the logic in (28). By definition, $t \in Q^2$ implies $\hat{k}(x^n(t), t) = \hat{k}(\gamma(t), t) = \arg\min_{k\in[k_1(t),k_2(t)]}[h(k, \gamma'(t))]$, therefore

$$\begin{aligned} h(\hat{k}(x^n(t), t), \gamma'(t)) &= h\left(\underset{k\in[k_1(t),k_2(t)]}{\arg\min}\,[h(k, \gamma'(t))], \gamma'(t)\right) \\ &\geqslant h\left(\underset{k\in[k_1(t),k_2(t)]}{\arg\min}\,[h(k, \gamma'(t))], \gamma'(t^{in})\right) - L\|\gamma''\|d \\ &\geqslant h\left(\underset{k\in[k_1(t^{in}),k_2(t^{in})]}{\arg\min}\,[h(k, \gamma'(t))], \gamma'(t^{in})\right) \\ &\quad - L\|\gamma''\|d - Ll\sqrt{\frac{V^2}{4} + 1}d \\ &\geqslant h\left(\underset{k\in[k_1(t^{in}),k_2(t^{in})]}{\arg\min}\,[h(k, \gamma'(t^{in}))], \gamma'(t^{in})\right) \\ &\quad - L\left(\|\gamma''\| + l\sqrt{\frac{V^2}{4} + 1}\right)d \\ &= h(k^2, \gamma'(t^{in})) - L\left(\|\gamma''\| + l\sqrt{\frac{V^2}{4} + 1}\right)d. \end{aligned} \tag{44}$$

Combining (44) with (43) yields

$$\int_{Q^2} h(\hat{k}(x^n(t),t),\alpha^n(t))\,dt \geqslant h(k^2,\gamma'(t^{in}))m(Q^2) - L\left(\|\gamma''\| + l\sqrt{\frac{V^2}{4}+1}\right)dm(Q^2). \tag{45}$$

Now we note that, by definition, $k^2 = k(t^{in})$, and that $h(k^i,\gamma'(t^{in})) \geqslant h(k(t^{in}), \gamma'(t^{in}))$ for $i = 1, 2, 3$. This allows us to combine (45) with (39)–(42) to obtain the analogue of (26):

$$\int_{t^{in}}^{t^{out}} h(\hat{k}(x^n(t),t),\alpha^n(t))\,dt \geqslant h(k(t^{in}),\gamma'(t^{in}))(t^{out}-t^{in}) - Cd^2 - 2L\varepsilon, \tag{46}$$

where, as before, $C = \|\gamma''\|L + Ll\sqrt{\frac{9V^2}{4}+1}$.

Next we construct, as before, a "cheap" path for $x^\varepsilon(t)$ where $\dot{x}^\varepsilon(t) = \alpha^\varepsilon(t)$, $x^\varepsilon(t^{in}) = x^n(t^{in})$, $x^\varepsilon(t^{out}) = x^n(t^{out})$, and $|\alpha^\varepsilon(t)| \leqslant V$. Since $k^\varepsilon(x,t) = \int_{-\varepsilon}^{\varepsilon} \hat{k}(x-y,t)\eta_\varepsilon(y)\,dy$, there must exist an x_0 where $|x_0| \leqslant \varepsilon$ and

$$|k^\varepsilon(\gamma(t^{in}) + x_0, t^{in}) - k(t^{in})| \leqslant l\varepsilon. \tag{47}$$

We want $x^\varepsilon(t)$ to move as quickly as possible from $(x^n(t^{in}), t^{in})$ to the curve $(\gamma(t) + x_0, t)$ (which runs parallel to the curve of discontinuity), stay with this curve as long as possible, and then move as quickly as possible to $(x^n(t^{out}), t^{out})$. This cost is bounded, as in (27), by

$$\int_{t^{in}}^{t^{out}} h(k^\varepsilon(x^\varepsilon(t),t),\alpha^\varepsilon(t))\,dt \leqslant \frac{8\varepsilon}{V}\|h\| + \int_{t^{in}}^{t^{out}} h(k^\varepsilon(\gamma(t)+x_0,t),\gamma'(t))\,dt. \tag{48}$$

Now we can exploit the fact the $\hat{k}(x,t)$ is Lipschitz continuous on either side of the discontinuity:

$$\begin{aligned}
&|k^\varepsilon(\gamma(t)+x_0,t) - k^\varepsilon(\gamma(t^{in})+x_0,t^{in})| \\
&\leqslant \int_{-\varepsilon}^{x_0} |\hat{k}(\gamma(t)+x_0-y,t) - \hat{k}(\gamma(t^{in})+x_0-y,t^{in})|\eta_\varepsilon(y)\,dy \\
&\quad + \int_{x_0}^{\varepsilon} |\hat{k}(\gamma(t)+x_0-y,t) - \hat{k}(\gamma(t^{in})+x_0-y,t^{in})|\eta_\varepsilon(y)\,dy \\
&\leqslant ld\sqrt{\frac{V^2}{4}+1}\int_{-\varepsilon}^{\varepsilon}\eta_\varepsilon(y)\,dy \\
&= ld\sqrt{\frac{V^2}{4}+1}.
\end{aligned} \tag{49}$$

From (47), (49), and the triangle inequality, we have that

$$|k^{\varepsilon}(\gamma(t)+x_0,t)-k(t^{in})| \leqslant ld\sqrt{\frac{V^2}{4}+1}+l\varepsilon,$$

which, combined with (48), gives

$$\begin{aligned}
&\int_{t^{in}}^{t^{out}} h(k^{\varepsilon}(x^{\varepsilon}(t),t),\alpha^{\varepsilon}(t))\,dt \\
&\leqslant \int_{t^{in}}^{t^{out}} h(k(t^{in}),\gamma'(t))\,dt + Lld^2\sqrt{\frac{V^2}{4}+1}+\left(Lld+\frac{8\|h\|}{V}\right)\varepsilon \\
&\leqslant \int_{t^{in}}^{t^{out}} h(k(t^{in}),\gamma'(t^{in}))\,dt + L\left(\|\gamma''\|+l\sqrt{\frac{V^2}{4}+1}\right)d^2+\left(Lld+\frac{8\|h\|}{V}\right)\varepsilon \\
&= h(k(t^{in}),\gamma'(t^{in}))(t^{out}-t^{in})+L\left(\|\gamma''\|+l\sqrt{\frac{V^2}{4}+1}\right)+d^2+\left(Lld+\frac{8\|h\|}{V}\right)\varepsilon.
\end{aligned} \tag{46}$$

As before, we subtract the costs of the $x^n(t)$ and $x^{\varepsilon}(t)$ paths in (46) and (50) to obtain the analogue of (30):

$$\int_{t^{in}}^{t^{out}} h(\hat{k}(x^n(t),t),\alpha^n(t))-h(k^{\varepsilon}(x^{\varepsilon}(t),t),\alpha^{\varepsilon}(t))\,dt \geqslant -C_1d^2-C_2\varepsilon-Lld\varepsilon, \tag{51}$$

where, as before, $C_1 \equiv C+L(\|\gamma''\|+l\sqrt{\frac{V^2}{4}+1})$, $C_2 \equiv 2L+\frac{8\|h\|}{V}$, and we note that neither of these constants depends on d,ε,δ, or n.

Just as in the previous subsection, we divide $[0,T]$ into three categories – $\tau^{\varepsilon},\tau^{\delta}$, and τ' (with the same definitions as before), and we choose $x^{\varepsilon}(t)=x^n(t)$ for all $t\in\tau^{\delta}\cup\tau'$, which, following the argument for (32)–(35), yields the analogue of (35):

$$\begin{aligned}
&\int_0^T h(\hat{k}(x^n(t),t),\alpha^n(t))-h(k^{\varepsilon}(x^{\varepsilon}(t),t),\alpha^{\varepsilon}(t))\,dt \\
&\quad\geqslant -2C_1(Td+Nd^2)-\left(2C_2\left(\frac{T}{d}+N\right)+LTl\right)\varepsilon \\
&\quad -\ 2Ll(T+Nd)\varepsilon-4\|h\|N\delta.
\end{aligned} \tag{52}$$

Since it must be the case that

$$u^{\varepsilon}(X,T) \leqslant \int_0^T h(k^{\varepsilon}(x^{\varepsilon}(t),t),\alpha^{\varepsilon}(t))\,dt+g(x^{\varepsilon}(T)) \tag{53}$$

and $x^\varepsilon(T) = x^n(T)$, we have from (38), (52), and (53) that

$$\begin{aligned} u(X,T) - u^\varepsilon(X,T) \geqslant & -2C_1(Td + Nd^2) \\ & - \left(2C_2\left(\frac{T}{d} + N\right) + Ll\right)\varepsilon - 2Ll(T + Nd)\varepsilon \\ & - 4\|h\|N\delta - \frac{1}{n}. \end{aligned} \tag{54}$$

Since this is true for any (arbitrarily large) n, we can drop the $\frac{1}{n}$ term at the end of (54). Now (as before) we take (54) and let ε shrink to zero (along a subsequence if necessary), which allows us to then consider infinitesimally small d, which allows us to then consider infinitesimally small δ, yielding the analogue to (37):

$$u(X,T) \geqslant \limsup_{\varepsilon\to 0}\ u^\varepsilon(X,T). \tag{55}$$

Combining (37) and (55) gives us the desired pointwise convergence:

$$u(X,T) = \lim_{\varepsilon\to 0}\ u^\varepsilon(X,T).$$

5. UNIFORM CONVERGENCE AND CONTINUITY OF THE SOLUTION

With the pointwise convergence of $u^\varepsilon \to u$ established in the previous section, we can next show this convergence is uniform and that u is Lipschitz continuous:

THEOREM 2. (1) *$u^\varepsilon \to u$ uniformly on any bounded subset of* Ω. (2) *u is Lipschitz continuous.*

Proof. Since we already know that $u^\varepsilon \to u$ pointwise from Theorem 1, both parts of the theorem follow from real analysis if we can establish that there exists a Lipschitz constant for u^ε that is independent of ε. We now proceed to show that this is the case, largely as a ramification of the fact that $\|h\|$ is independent of ε.

Consider two points (X_1, T_1) and (X_2, T_2). Let $x_1^\varepsilon(t)$ be a path over the domain $t \in [0, T_1]$ where $x_1^\varepsilon(0) = X_1$ and $|\dot{x}_1^\varepsilon(t)| \leqslant \hat{V}$. Let $x_2^\varepsilon(t)$ be a path over the domain $t \in [0, T_2]$ where $x_2^\varepsilon(0) = X_2$ and $|\dot{x}_2^\varepsilon(t)| \leqslant V$. Define $r = \sqrt{(X_1 - X_2)^2 + (T_1 - T_2)^2}$. Since $V \geqslant 2\hat{V}$, we know that if r is sufficiently small, $x_2^\varepsilon(t)$ can be chosen to intersect any $x_1^\varepsilon(t)$. We choose $x_2^\varepsilon(t)$ to intersect $x_1^\varepsilon(t)$ as quickly as possible and then to follow $x_1^\varepsilon(t)$. From geometric arguments, it can be seen that the amount of time $x_2^\varepsilon(t)$ requires before the intersection occurs is, at most, $\left((\frac{1}{\hat{V}} + \hat{V})/\sqrt{\hat{V}^2 + 1}\right)r$ and the amount of time

$x_1^\varepsilon(t)$ can propagate without being intersected is, at most, $\left((\frac{1}{2\hat{V}}+\hat{V})/\sqrt{\hat{V}^2+\frac{1}{4}}\right)r$.

Now let $x_{1,n}^\varepsilon(t)$ be a minimizing sequence, i.e.

$$u^\varepsilon(X_1,T_1)\geqslant \int_0^{T_1} h(k^\varepsilon(x_{1,n}^\varepsilon(t),t),\dot{x}_{1,n}^\varepsilon(t))\,dt+g(x_{1,n}^\varepsilon(T_1))-\frac{1}{n}.$$

(Note that this is only possible because of the assumption that the characteristic speeds are bounded by $\hat{V}$.) If we define T^n as the first time when $x_{1,n}^\varepsilon(t)$ is intersected by $x_{2,n}^\varepsilon(t)$ (i.e. $x_{1,n}^\varepsilon(T^n)=x_{2,n}^\varepsilon(T^n-(T_1-T_2))$), then we have that

$$\begin{aligned}u^\varepsilon(X_1,T_1)\geqslant &-\|h\|\frac{\frac{1}{2\hat{V}}+\hat{V}}{\sqrt{\hat{V}^2+\frac{1}{4}}}r\\ &+\int_{T^n}^{T_1} h(k^\varepsilon(x_{1,n}^\varepsilon(t),t),\dot{x}_{1,n}^\varepsilon(t))\,dt+g(x_{1,n}^\varepsilon(T_1))-\frac{1}{n}\end{aligned}$$

and

$$u^\varepsilon(X_2,T_2)\leqslant\|h\|\frac{\frac{1}{\hat{V}}+\hat{V}}{\sqrt{\hat{V}^2+1}}r+\int_{T^n}^{T_1} h(k^\varepsilon(x_{1,n}^\varepsilon(t),t),\dot{x}_{1,n}^\varepsilon(t))\,dt+g(x_{1,n}^\varepsilon(T_1)).$$

Subtracting these two inequalities yields

$$u^\varepsilon(X_2,T_2)-u^\varepsilon(X_1,T_1)\leqslant\left(\frac{\frac{1}{\hat{V}}+\hat{V}}{\sqrt{\hat{V}^2+1}}+\frac{\frac{1}{2\hat{V}}+\hat{V}}{\sqrt{\hat{V}^2+\frac{1}{4}}}\right)\|h\|r+\frac{1}{n}$$

and letting $n\to\infty$, we have that

$$u^\varepsilon(X_2,T_2)-u^\varepsilon(X_1,T_1)\leqslant kr,$$

where $k=\left((\frac{1}{\hat{V}}+\hat{V})/\sqrt{\hat{V}^2+1}\right)+\left((\frac{1}{2\hat{V}}+\hat{V})/\sqrt{\hat{V}^2+\frac{1}{4}}\right)\|h\|\leqslant 2\|h\|\left(\frac{1}{\hat{V}^2}+1\right)$. Repeating this argument with the roles of (X_1,T_1) and (X_2,T_2) switched yields

$$u^\varepsilon(X_1,T_1)-u^\varepsilon(X_2,T_2)\leqslant kr$$

and so we have that

$$|u^\varepsilon(X_1,T_1)-u^\varepsilon(X_2,T_2)|\leqslant kr,$$

therefore k is a Lipschitz constant that is independent of ε. ∎

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