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ON THE EARLY EXERCISE BOUNDARY OF THE AMERICAN PUT OPTION*

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Abstract. We study the short time behavior of the early exercise boundary for American style put options in the Black–Scholes theory. We develop an asymptotic expansion which shows that the simple lower bound of Barles et al. is a more accurate approximation to the actual boundary than the more complex upper bound. Our expansion is obtained through iteration using a boundary integral equation. This integral equation is derived from the time derivative of the option value function, which closely resembles the classical Stefan free boundary value problem for melting ice. Our analytical results are supported by numerical computations designed for very short times.

Key words. Black–Scholes, free boundary, American put

AMS subject classifications. 35K20, 35K60, 35R35, 91B28

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1. Introduction.

The key to determining the value of an American put option is finding the "early exercise boundary," which specifies the conditions under which the option should be exercised before it expires. Unfortunately, the early exercise boundary cannot be solved in closed form, and, as a result, the option's value cannot be solved in closed form either. In this paper, however, we will be able to determine a relatively simple integral equation for the early exercise boundary by exploiting the especially strong connections between the Black–Scholes equation governing *the derivative* with respect to time of the option's value and the classical Stefan problem, which governs melting ice. This integral equation is suitable for iterationlike nonlinear Volterra equations of the second kind—which it closely resembles—and also will be used to find an analytic approximation to the early exercise boundary when the option is close to expiration. Our analytic approximation will be consistent with the rigorous bounds determined by Barles et al. in [1] but not entirely consistent with the work of Kuske and Keller in [7] as will be explained below. Our results are consistent with the more recent independent work of Chen, Chadam, and Stamicar [2] and Evans, Kuske, and Keller [3].

We begin with the Black–Scholes theory [8] for the value of an American put option, $f(S,t)$, where S is the price of the stock underlying the option and t is the time remaining until the option expires. The American—as opposed to European—put option may be exercised at any time for a yield of $K-S$, where K is the strike price of the option. This leads to our splitting the domain $\{S \in (-\infty,\infty),\, t \in [0,\infty)\}$ into two regions which are separated by the early exercise boundary, $S = b(t)$: If $S \le b(t)$, then the option should be immediately exercised and, therefore, $f(S,t) = K - S$; if $S > b(t)$, then the option should not be exercised, and $f(S,t)$—which is greater than

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$K - S$—is governed by the Black–Scholes parabolic partial differential equation

$$f_t = \frac{\sigma^2 S^2}{2} f_{SS} + rSf_S - rf, \tag{1.1}$$

where σ is the stock's volatility and r is the risk-free interest rate. At $t = 0$ the option must be exercised or discarded so $f(S,0) = \max(K - S, 0)$. The final fact we need, which is possibly less intuitive but easy to establish mathematically [8], is that $f_S(b(t),t) = -1$ for $t > 0$; i.e., f is C^1 smooth on the early exercise boundary.

There are useful parallels between the Black–Scholes equation for f and the Stefan problem for the temperature of water bordering a melting block of ice. The equations for both f and the water temperature are parabolic partial differential equations and, more importantly, both contain "free boundaries," that is, boundaries that are not explicitly known, which correspond to the early exercise boundary, $b(t)$, in the Black–Scholes case and the location of the interface between the water and the ice in the Stefan problem. While these parallels can be exploited to find integral equations for $b(t)$ (see [7]), we will show that the parallels between the formulation of the Black–Scholes equation for the time derivative, f_t, and the Stefan problem are even stronger, since, in addition to their both being free boundary parabolic partial differential equation problems, their conditions on the free boundary and at $t = 0$ also heavily parallel each other. This will allow us to adapt standard techniques for the Stefan problem (see, e.g., [4]) to determine the following integral representation for $f_t(S,t)$ in the region $S > b(t)$:

$$f_t(S,T) = \hat{G}\left(\ln\left(\frac{S}{K}\right), t\right) + \frac{2r}{\sigma^2}\int_0^t \frac{\dot{b}(s)}{b(s)}\hat{G}\left(\ln\left(\frac{S}{b(s)}\right), t-s\right) ds, \tag{1.2}$$

where $\hat{G}$ is the following Green's function:

$$\hat{G}(X,T) = \frac{K\sigma}{\sqrt{8\pi T}} \exp\left[-\frac{\left(X - \left(r - \frac{\sigma^2}{2}\right)T\right)^2}{2\sigma^2 T} - rT\right].$$

Delicate problems may arise when letting $S \to b(t)$ in equations similar to (1.2); however, we will show that these problems do not arise for (1.2), allowing us to easily obtain our integral equation for $b(t)$, which is

$$\hat{G}\left(\ln\left(\frac{b(t)}{K}\right), t\right) = -\frac{2r}{\sigma^2}\int_0^t \frac{\dot{b}(s)}{b(s)}\hat{G}\left(\ln\left(\frac{b(t)}{b(s)}\right), t-s\right) ds. \tag{1.3}$$

(Note that in the next section (1.3) is rewritten as (2.20), which has a more simple form due to the use of scaled variables. We use unscaled variables in this section, which are more useful for applications, and introduce scaled variables in the next section, which are more useful mathematically. Both types of variables commonly appear in the literature.) The closest equation to (1.3) that we have found in the published literature is by Hou, Little, and Pant (see equation (3.11) in [5]). Both the Hou expression and ours relate $b(t)$ to a single integral; their expression is more complex, but it has the advantage of not containing $\dot{b}(t)$. We have also recently become aware that (1.3) has been discovered independently by Chen, Chadam, and Stamicar in the preprint [2].

We next consider the nature of $b(t)$ when t is small. It is easy to establish that $b(t) < K$ and $b(t) \to K$ as $t \to 0$. Our derivation of the free boundary conditions for f_t will also establish that

$$f_{SS}(b(t), t) = \frac{2Kr}{\sigma^2 b^2(t)}. \tag{1.4}$$

Informally, (1.4) can be exploited to quickly give a rough estimate of $b(t)$ for small t by heuristically supposing that f_{SS} is a simple Gaussian[1]

$$f_{SS} \approx \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-(S-K)^2/\left(2\sigma^2 K^2 t\right)}, \tag{1.5}$$

and, letting $S = b(t)$ in (1.5), we set (1.5) equal to (1.4), yielding

$$b(t) \approx K\left(1 - \sigma\sqrt{-t\ln\left(\frac{8\pi r^2}{K^2\sigma^2}t\right)}\right).$$

While we use (1.4) to obtain a far more rigorous and complete analysis of $b(t)$ in the next section, we note that the above approximation is quite close to the short time bounds discovered by Barles et al. in [1]; that is,

$$b(t) = K\left(1 - \sigma\sqrt{-t\ln\left(Ctv^2(t)\right)}\right), \tag{1.6}$$

where $C = \frac{8\pi r^2}{\sigma^2}$ and

$$\frac{1}{4} \leq \lim_{t\to 0} v^2(t) \quad \text{and} \quad \lim_{t\to 0} \frac{v^2(t)}{-\ln(Ct)} \leq 1.$$

It is difficult to discern v from numerical experiments since t must be extremely small for the true nature of the logarithmic terms to become clear. Our results will show that

$$v \approx 1 + \frac{1}{\ln(Ct)} + O\left(\left(\frac{1}{\ln(Ct)}\right)^2\right) \tag{1.7}$$

and, therefore, the lower bound is closer. This expression has also been independently obtained by Chen, Chadam, and Stamicar in the preprint [2]. Also independent of our work is a recent preprint by Evans, Kuske, and Keller [3] which corrects an earlier assertion by Kuske and Keller in [7] that the upper bound is closer. Finally, we point out the recent work of Knessl in [6], which analyzes the expansion for v using Laplace transform methods.

The next section primarily details the Black–Scholes equation for f_t and how it leads to (1.2) and (1.3). In section 3, we use this integral equation to obtain the asymptotic expression (1.7). The final section contains computational evidence supporting this asymptotic result.

[1]Since $f(S,0)$ is a ramp, $f_S(S,0)$ is a step and $f_{SS}(S,0)$ is a δ function centered at $S = K$. The diffusion coefficient at $S = K$ is $\sigma^2 S^2 = \sigma^2 K^2$. In light of the fact that $f_{SS}(S,0) = \delta(S-K)$, one might expect that $\lim_{t\to 0} f_{SS}(b(t),t) = 0$ or ∞, but note from (1.4) that $\lim_{t\to 0} f_{SS}(b(t),t) = \frac{2r}{\sigma^2 K}$!

2. Derivation of integral and asymptotic equations for the early exercise boundary. We begin with (1.1), the Black–Scholes equation for f in the region $S > b(t)$, together with f's boundary conditions

$$f(b(t), t) = K - b(t), \tag{2.1}$$

$$f_S(b(t), t) = -1 \tag{2.2}$$

and f's initial condition

$$f(S, 0) = \max(K - S, 0). \tag{2.3}$$

The characterization of f_t is similar. Clearly the partial differential equation is not changed:

$$(f_t)_t = \frac{\sigma^2}{2} S^2 (f_t)_{SS} + rS (f_t)_S - r (f_t). \tag{2.4}$$

We need to find two boundary conditions to replace (2.1) and (2.2). Differentiating (2.1) with respect to t gives

$$f_S \dot{b} + f_t = -\dot{b}.$$

Using (2.2), this simplifies to

$$f_t(b(t), t) = 0. \tag{2.5}$$

This is the first of the two desired conditions. Substituting (2.5) into (1.1) gives, at $S = b$,

$$0 = \frac{\sigma^2}{2} b^2 f_{SS} + rbf_S - rf.$$

This, together with (2.1) and (2.2), gives

$$f_{SS}(b(t), t) = \frac{2Kr}{\sigma^2 b^2(t)}, \tag{2.6}$$

which was exploited in the introduction to find a rough estimate of $b(t)$. Finally, differentiating (2.2) with respect to t,

$$f_{SS} \dot{b} + f_{St} = 0,$$

and using (2.6) gives the second boundary condition

$$(f_t)_S (b(t), t) = -\frac{2Kr}{\sigma^2 b^2(t)} \dot{b}(t). \tag{2.7}$$

For the purpose of deriving the integral equation, the initial data for f_t will be

$$f_t(S, 0) = \frac{\sigma^2 K^2}{2} \delta(S - K) \tag{2.8}$$

for the following reason: As will become clear below, we are interested only in $\int_{b(t)}^{\infty} f_t(S, t) dS$ as $t \to 0$. We can analyze this integral using (1.1) by noting first that $f_S(S, t)$ and $f(S, t)$ are bounded and go to zero as $t \to 0$ for any $S > K$. While

$f_{SS}(S,t)$ also goes to zero as $t \to 0$ for any $S > K$, $f_{SS}(S,t)$ is not bounded as $t \to 0$ at $S = K$. Rather, $f_S(b(t),t) = -1$ so, for any small $\varepsilon > 0$, $\int_{b(t)}^{K+\varepsilon} f_{SS}(S,t)dS \to 1$ as $t \to 0$. It is in this sense that we say that $f_{SS}(S,0) = \delta(S-K)$, and therefore, applying (1.1) , we have (2.8) .

The analogy of the two boundary conditions for f_t to those that arise in the physical Stefan problem is close. In the physical Stefan problem we specify the temperature at the boundary where the ice is melting. This is close to (2.5). We also relate the speed of the boundary motion, which is proportional to the rate at which ice melts, to the gradient of the temperature at the boundary. This is similar to (2.7).

It is more convenient to work in traditional normalized coordinates with the stock price variable normalized by the strike price,

$$S = Ke^x, \tag{2.9}$$

and the time variable normalized by the volatility,

$$t = \frac{2}{\sigma^2}\tau.$$

The early exercise boundary now occurs at $x = \beta$, where

$$\beta = \ln\left(\frac{b}{K}\right),$$

and we will denote the unknown in these new x and τ variables by $u(x,\tau)$:

$$u(x,\tau) = \frac{2}{K\sigma^2} f_t(S,t) = \frac{2}{K\sigma^2} f_t\left(Ke^x, \frac{2}{\sigma^2}\tau\right).$$

In these new variables, the partial differential equation (1.1) takes the simplified constant coefficient form

$$u_\tau = u_{xx} + (\rho - 1)u_x - \rho u, \tag{2.10}$$

where ρ is a nondimensional measure of the risk-free rate:

$$\rho = \frac{2r}{\sigma^2}.$$

The boundary condition (2.5) becomes

$$u(\beta,\tau) = 0. \tag{2.11}$$

To translate the other boundary condition (2.7), use $\dot{b}/b = \dot{\beta}$. Also, $\dot{\beta} = \frac{d\beta}{dt} = \frac{\sigma^2}{2}\frac{d\beta}{d\tau}$, so the right side of (2.7) is

$$\frac{-Kr}{b}\beta',$$

where β' denotes $\frac{d\beta}{d\tau}$ (which we will use throughout the rest of this paper). Since $(f_t)_S = \frac{1}{S}(f_t)_x$ and we have $S = b$ at the early exercise boundary, (2.7) becomes simply

$$u_x(\beta,\tau) = -\rho\beta'. \tag{2.12}$$

Finally, the initial condition (2.8) translated into our new variables is

$$u(x,0) = \delta(x). \tag{2.13}$$

This completes the formulation of the problem in dimensionless variables.

The standard method for finding an integral equation for the free boundary (see, e.g., [4]) starts with the Green's function for (2.10),

$$G(X,T) \equiv \frac{1}{\sqrt{4\pi T}} \exp\left[-\frac{(X+(\rho-1)T)^2}{4T} - \rho T\right],$$

which satisfies

$$G_T = G_{XX} + (\rho-1)G_X - \rho G. \tag{2.14}$$

(Note that G is proportional to $\hat{G}$, the Green's function used in the introduction once we transform to the original variables.) If there were no boundary, we would have, for any $s \in [0,\tau)$,

$$u(x,\tau) = \int_{-\infty}^{\infty} G(x-y,\tau-s)u(y,s)dy, \tag{2.15}$$

so if we set $s=0$, we can describe $u(x,\tau)$ in terms of its initial condition.

For our free boundary problem we consider the analogue of the integral in (2.15):

$$I(x,\tau,s) = \int_{\beta(s)}^{\infty} G(x-y,\tau-s)u(y,s)dy. \tag{2.16}$$

Note that as $s \to \tau$, $G(x-y,\tau-s)$ converges to a Gaussian function whose variance shrinks to zero; i.e., $\lim_{s\to\tau} G(x-y,\tau-s) = \delta(x-y)$ and therefore for any $x > \beta(\tau)$

$$\lim_{s\to\tau} I(x,\tau,s) = u(x,\tau). \tag{2.17}$$

With this in mind it is not surprising that we can relate the solution, $u(x,\tau)$, to the initial condition by integrating $I_s(x,\tau,s)$ between $s=0$ and $s=\tau$.

We first differentiate (2.16), yielding

$$\begin{aligned} I_s(x,\tau,s) = &-\beta'(s)G(x-\beta(s),\tau-s)u(\beta(s),s) \\ &-\int_{\beta(s)}^{\infty} G_T(x-y,\tau-s)u(y,s)dy \\ &+\int_{\beta(s)}^{\infty} G(x-y,\tau-s)u_s(y,s)dy. \end{aligned}$$

The first term on the right vanishes because of (2.11). In the third integral, substitute (2.10) , integrate by parts to put all the derivatives on G, and then use (2.14). Three terms evaluated on the boundary remain. Two vanish by the boundary condition (2.11), and we apply the other boundary condition (2.12) to the remaining term to obtain

$$I_s(x,\tau,s) = \rho G(x-\beta(s),\tau-s)\beta'(s). \tag{2.18}$$

Next we look at $I(x, \tau, 0)$. From the form of (2.16) it is now clear why the initial condition (2.8) was formulated using the integral from the early exercise boundary to infinity. This allows us to now substitute (2.13) into (2.16), yielding

$$I(x, \tau, 0) = G(x, \tau).$$

Now we can integrate $I_s(x, \tau, s)$ from $s = 0$ to $s = \tau$ to obtain

$$u(x, \tau) = G(x, \tau) + \rho \int_0^\tau \beta'(s) G(x - \beta(s), \tau - s) ds, \tag{2.19}$$

which corresponds to (1.2) in the introduction. We stress that (2.19) has only been established for $x > \beta(\tau)$. If $x = \beta(\tau)$, then we see that as we try to proceed from (2.16) to (2.17) we integrate only over *half* of the Gaussian, which means that if $x = \beta(\tau)$, a factor of $1/2$ must be inserted in front of u on the left-hand side of (2.19) for the equation to be valid. This ends up being unimportant for us since $u(\beta(\tau), \tau) = 0$, so, letting $x = \beta(\tau)$, we have from (2.19) that

$$G(\beta(\tau), \tau) = -\rho \int_0^\tau \beta'(s) G(\beta(\tau) - \beta(s), \tau - s)\, ds. \tag{2.20}$$

Expression (2.20)—which corresponds to expression (1.3) if we transform back to our original variables—is similar to a Volterra equation of the second kind. Specifically, like Volterra equations of the second kind, we can try to iteratively solve for $\beta(\tau)$ by inserting an initial expression for $\beta(\tau)$ into the integral on the right-hand side of (2.20), which leads to a new expression for $\beta(\tau)$ on the left-hand side. We then insert the new $\beta(\tau)$ into the integral on the right-hand side and repeat the process as many times as is necessary.

3. Short time asymptotics. By translating the Barles result (1.6) for the short time asymptotics of the early exercise boundary into our new coordinates, we have that

$$\beta(\tau) = -\sqrt{-2\tau \ln(v^2 c\tau)}, \tag{3.1}$$

where $c = 4\pi\rho^2$ and

$$\frac{1}{4} \le \lim_{\tau \to 0} v^2(\tau) \quad \text{and} \quad \lim_{\tau \to 0} \frac{v^2(\tau)}{-\ln(c\tau)} \le 1.$$

We will now use the iteration algorithm described above to determine the nature of v: Letting $v_0 = 1$ be our initial guess for v, we will insert (3.1) with $v = v_0$ into the right-hand side of (2.20) which will yield a value for v from the left-hand side of (2.20) which we define to be v_1. Then we insert (3.1) with $v = v_1$ into the right-hand side of (2.20) to yield v_2 and repeat this process to establish a sequence $v_0, v_1, v_2, v_3, \ldots$ which converges to v. In this fashion we will establish that

$$v = 1 + \frac{1}{\ln(c\tau)} + O\left(\left(\frac{1}{\ln(c\tau)}\right)^2\right), \tag{3.2}$$

which is equivalent to (1.7) from the introduction.

Our analysis begins with the left-hand side (LHS) of (2.20). Inserting (3.1) into the left-hand side (LHS) of (2.20) yields

$$\text{LHS} = G\left(\beta(\tau), \tau\right) = \frac{1}{\sqrt{4\pi\tau}} \exp\left[-\frac{\left(-\sqrt{-2\tau \ln\left(v^2 c\tau\right)} + (\rho-1)\tau\right)^2}{4\tau} - \rho\tau \right]. \tag{3.3}$$

As τ gets smaller, the $(\rho - 1)\tau$ and $\rho\tau$ terms in (3.3) are dominated by the $\sqrt{-2\tau \ln\left(v^2 c\tau\right)}$ term, so

$$\text{LHS} \approx \frac{1}{\sqrt{4\pi\tau}} \exp\left[-\frac{\left(-\sqrt{-2\tau \ln\left(v^2 c\tau\right)}\right)^2}{4\tau} \right] \qquad \text{for } \tau \text{ small.}$$

Algebraic simplification of this expression yields

$$\text{LHS} \approx \frac{v\sqrt{c}}{\sqrt{4\pi}} \qquad \text{for } \tau \text{ small,}$$

and therefore, since $c = 4\pi\rho^2$, we have

$$\text{LHS} \approx v\rho \qquad \text{for } \tau \text{ small.} \tag{3.4}$$

Now we consider the right-hand side of (2.20). The primary contribution to the integral representing the right-hand side (RHS) of (2.20) comes from the values of s near τ (no matter what v is), so we consider $\beta(s)$ and $\beta'(s)$ as Taylor polynomials around $s = \tau$:

$$\begin{aligned}
\text{RHS} &= -\rho \int_0^\tau \beta'(s) G\left(\beta(\tau) - \beta(s), \tau - s\right) ds \\
&\approx -\rho \int_0^\tau \left[\beta'(\tau) - \beta''(\tau)(\tau - s) + \text{h.o.t.}\right] \\
&\quad \times G\left(\beta'(\tau)(\tau - s) - \beta''(\tau)\frac{(\tau - s)^2}{2} + \text{h.o.t.}, \tau - s\right) ds \\
&\approx \frac{-\rho}{\sqrt{4\pi}} \int_0^\tau \left[\beta'(\tau) - \beta''(\tau)(\tau - s) + \text{h.o.t.}\right] \frac{1}{\sqrt{(\tau - s)}} \\
&\quad \times \exp\left[-\frac{\left(\begin{array}{c} \beta'(\tau)(\tau - s) - \beta''(\tau)\frac{(\tau-s)^2}{2} + \text{h.o.t.} \\ +(\rho - 1)(\tau - s) \end{array}\right)^2}{4(\tau - s)} - \rho(\tau - s) \right] ds,
\end{aligned} \tag{3.5}$$

where h.o.t. stands for "higher order terms." As with the analysis of the left-hand side, the $(\rho - 1)(\tau - s)$ and $\rho(\tau - s)$ terms in the exponential will be much smaller than the other terms as $\tau \to 0$, so we can neglect them. Neglecting these terms and substituting

$$x = -\beta'(\tau)\sqrt{\frac{\tau - s}{2}}$$

into (3.5) yields

$$\text{RHS} \approx \sqrt{\frac{2}{\pi}}\rho \int_0^{-\beta'(\tau)\sqrt{\frac{\tau}{2}}} \left(1 - 2Dx^2 + \text{h.o.t.}\right) \exp\left[-\frac{1}{2}\left(x - Dx^3 + \text{h.o.t.}\right)^2\right] dx, \tag{3.6}$$

where $D = \frac{\beta''(\tau)}{(\beta'(\tau))^3}$. Even though (as we will see) $\lim_{\tau\to 0} -\beta'(\tau)\sqrt{\frac{\tau}{2}} = \infty$, the primary contribution to (3.6) comes from the values of x near 0. So, since $\lim_{\tau\to 0} D = 0$ (which we will also see), we have that

$$\begin{aligned} \text{RHS} &\approx \sqrt{\frac{2}{\pi}}\rho \int_0^{-\beta'(\tau)\sqrt{\frac{\tau}{2}}} \left(1 - 2Dx^2 + \text{h.o.t.}\right) \exp\left[-\frac{x^2}{2}\right] \exp\left[Dx^4 + \text{h.o.t.}\right] dx \\ &\approx \sqrt{\frac{2}{\pi}}\rho \int_0^{\infty} \left(1 - 2Dx^2 + \text{h.o.t.}\right) \exp\left[-\frac{x^2}{2}\right] \left(1 + Dx^4 + \text{h.o.t.}\right) dx, \end{aligned}$$

and so

$$\begin{aligned} \text{RHS} &\approx \sqrt{\frac{2}{\pi}}\rho \int_0^{\infty} \left(1 - 2Dx^2 + Dx^4 + \text{h.o.t.}\right) \exp\left[-\frac{x^2}{2}\right] dx \\ &= \rho(1 - 2D + 3D + \text{h.o.t.}) \\ &= \rho(1 + D + \text{h.o.t.}). \end{aligned} \tag{3.7}$$

So, combining (3.4) and (3.7), we have

$$v \approx 1 + D + \text{h.o.t.}, \tag{3.8}$$

which is our desired recurrence relation.

Now we set $v = v_0 = 1$ in (3.1), so we have

$$\begin{aligned} \beta(\tau) &\approx -\sqrt{-2\tau \ln(c\tau)}, \\ \beta'(\tau) &\approx \frac{\ln(c\tau) + 1}{\sqrt{-2\tau \ln(c\tau)}}, \\ \beta''(\tau) &\approx \frac{\ln^2(c\tau) + 1}{(-2\tau \ln(c\tau))^{\frac{3}{2}}}, \end{aligned}$$

and so

$$D \approx \frac{\ln^2(c\tau) + 1}{(\ln(c\tau) + 1)^3} \approx \frac{1}{\ln(c\tau)}.$$

(Note that $\lim_{\tau\to 0} -\beta'(\tau)\sqrt{\frac{\tau}{2}} = \infty$ and $\lim_{\tau\to 0} D = 0$ as indicated previously.) If we ignore the h.o.t., we have from (3.8) that $v_1 = 1 + \frac{1}{\ln(c\tau)}$. Setting $v = v_1$ in (3.1) , and repeating these calculations, again shows that $D \approx \frac{1}{\ln(c\tau)}$, and so, since this will be the case for as many iterations as we want, we have that

$$v = 1 + \frac{1}{\ln(c\tau)} + \text{h.o.t.}$$

Unfortunately, we can easily show that h.o.t. $= O((\frac{1}{\ln(c\tau)})^2)$, so these higher order terms do influence the value of v unless τ is extremely small.

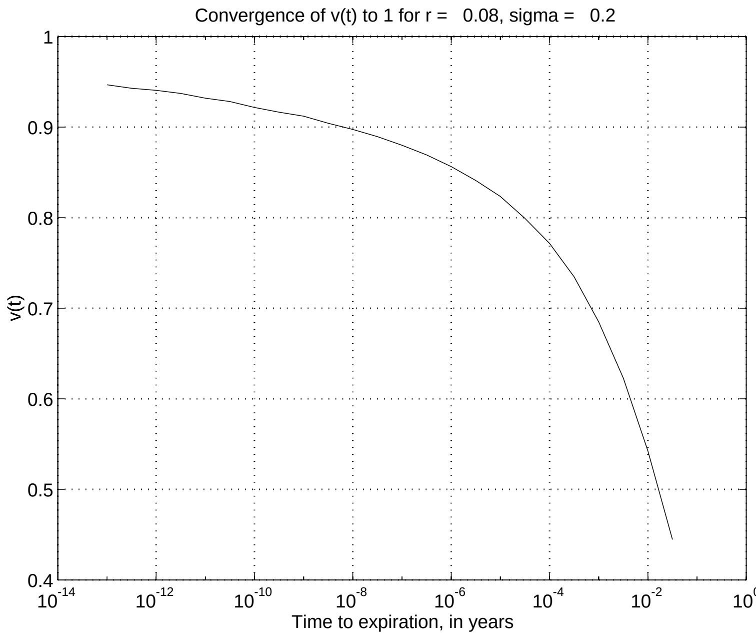

FIG. 4.1. *In Figures* 4.1–4.3*, for all three r and σ combinations, we see that $v(t)$ appears to converge to* 1 *as t approaches* 0 *even though the behaviors for the three combinations differ at larger times.*

4. Computational methods and results. We computed the free boundary using a more or less standard method, with details chosen to minimize the effect of roundoff error in short time computation. To compute the free boundary near 0 rather than near K, we use, as in (2.9), $S = Ke^x$ transforming the Black–Scholes equation (1.1) into the constant coefficient form

$$F_t = \frac{\sigma^2}{2} F_{xx} + \left(r - \frac{\sigma^2}{2} \right) F_x - rF$$

with

$$F(x,t) = \frac{1}{K} f(S,t),$$

where the initial condition (2.3) is

$$F(x,0) = F_0(x) \equiv \max(1 - e^x, 0).$$

Note that since the American put can be exercised at any time, we always have the intrinsic constraint

$$F(x,t) \geq F_0(x). \tag{4.1}$$

To avoid roundoff problems in computing $1 - e^x$, we use only the built in exponential function in double precision for $|x| > 10^{-4}$; for $|x| \leq 10^{-4}$, we use the Horner form of the Taylor polynomial

$$1 - e^x \approx -x \left(1 + \frac{x}{2} \left(1 + \frac{x}{3} \left(1 + \frac{x}{4} \right) \right) \right)$$

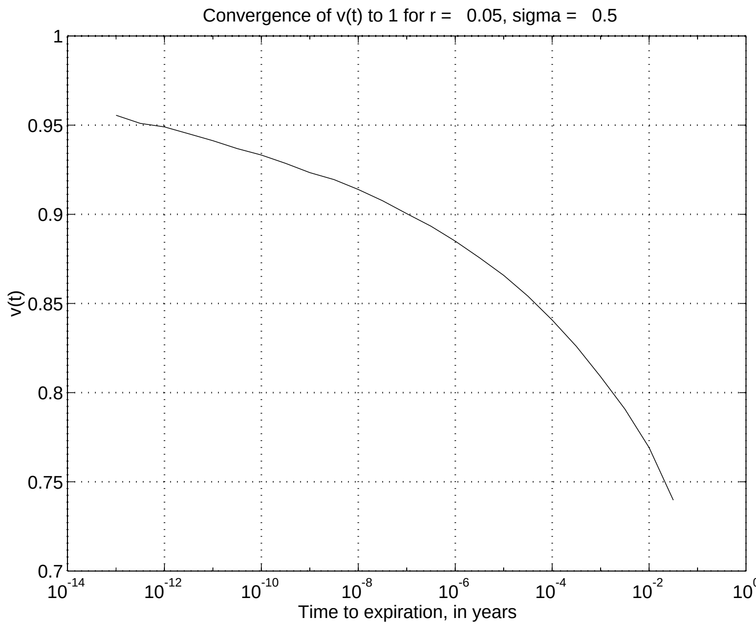

FIG. 4.2.

to reduce roundoff error.

Each plotted value of $v(t)$ in Figures 4.2–4.3 is produced by performing a simulation up to time t and recording the final v value. A single run could not be used to determine all of the data points at once because the value of v produced in the early stages of any specific run (that is, for the first few iterations in time, regardless of the value of Δt) was found not to be accurate for our purposes. Given a specific stopping time, t, we chose $x_{\min}$ and $x_{\max}$, the lower and upper bounds for the x domain, by using formulas roughly motivated by the theory

$$x_{\min} = -\frac{3}{2}\sigma\sqrt{-t\ln(t)}, \qquad x_{\max} = 4\sigma\sqrt{t}.$$

We have checked that the lower bound, $x_{\min}$, is irrelevant in that $\beta(t)$ never reached $x_{\min}$ in any of our computations. We also checked that varying $x_{\max}$ (say, using $x_{\max} = 5\sigma\sqrt{t}$) has a negligible effect on the solution. The x domain was divided into nx uniformly spaced grid points (therefore $\Delta x = (x_{\max} - x_{\min})/(nx-1)$), and a corresponding time step of $\Delta t = CFL \cdot \Delta x^2$ was chosen, where the CFL constant was 0.9 for all of our runs. We used $nx = 8000$ for the data presented in Figures 4.2–4.3, which agree, to beyond plotting accuracy, with data we obtained using $nx = 16,000$.

The numerical approximations were computed by the forward Euler/trinomial tree method taking into account the intrinsic constraint (4.1):

$$F_k^{n+1} = \max\left(\widetilde{F}_k^{n+1}, F_0(x)\right),$$

where

$$\widetilde{F}_k^{n+1} = F_k^n + \frac{\sigma^2}{2}\frac{\Delta t}{\Delta x^2}\left(F_{k+1}^n - 2F_k^n + F_{k-1}^n\right) + \left(r - \frac{\sigma^2}{2}\right)\frac{\Delta t}{2\Delta x}\left(F_{k+1}^n - F_{k-1}^n\right) - r\Delta t F_k^n. \tag{4.2}$$

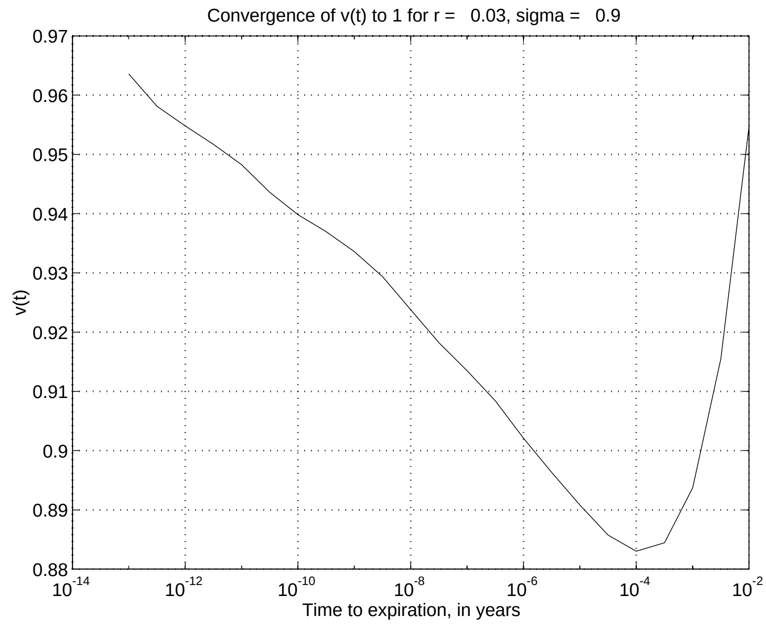

FIG. 4.3.

At each time step, n, the early exercise boundary is approximated by $\beta_n = x_{k^*_n}$, the smallest grid point $x_k = x_{\min} + k\Delta x$, where the intrinsic constraint (4.1) is still binding; that is, in theory,[2]

$$k^n_* = \max\left(k \mid F^n_k = F_0(x_k)\right). \tag{4.3}$$

By using the approximation of the early exercise boundary at the end of the run, $\beta(t)$, we can compute $v(t)$ by inverting (3.1) to obtain

$$v(t) = \frac{\sigma}{\sqrt{8\pi r^2 t}} \exp\left(-\frac{\beta^2(t)}{2\sigma^2 t}\right).$$

This method appears to produce reliable results for $v(t)$ for t as small as 10^{-13}.

Figures 4.2–4.3 contain plots of computed values for $v(t)$ over a wide range of t values with three different sets of values for r and σ. Our theoretical analysis in (3.2) indicates that $v \to 1$ as $t \to 0$. From the figures, we see that this is born out in computations for t as small as 10^{-13}. We consider our computation for smaller t values unreliable because of computer roundoff error, most likely due to the cancellations within the second derivative approximation in (4.2). From the figures it is clear that our computations do not support $v(t)$ being closer to Barles's upper bound, $\sqrt{-\ln(Ct)}$, since this bound implies that $v(t) \to \infty$ as $t \to 0$.

The C++ and Matlab programs used in this work may be downloaded from http://www.math.nyu.edu/faculty/goodman/research/research.html.

[2]Actually, the formula (4.3) gave trouble with computer arithmetic for small t. It was more reliable to use $k^n_* = \max\left(k \mid \widetilde{F}^n_k < F_0(x_k)\,(1+\varepsilon)\right)$ with $\varepsilon = 10^{-15}$.

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