Math 13 Practice Problems

These problems were compiled by Ed Schaefer. They should not be interpreted to indicate what will be on my tests in any way, but they can give you some practice working on new problems. Unlike an exam, I have ordered the problems so that, to a large degree, they follow the order of the class. This allows you to stop when you encounter material we haven't covered yet. Some problems have hints. If you need the hints, they are between the questions and the answers.

Questions:

1. Compute $\frac{d}{dx}3\sinh(3/x)$
2. Simplify $\cosh(\ln(2))$.
3. Find the sum of the series $\sum_{n=0}^{\infty}\frac{1}{n!}$.
4. Find the sum of all of the convergent series

   $$a.\ \sum_{n=0}^{\infty}\frac{\sin(\frac{\pi}{2}n)}{3^{n+1}} \qquad b.\ \sum_{n=1}^{\infty}\frac{-2}{n(n+1)} \qquad c.\ \sum_{n=0}^{\infty}\frac{(-25)^n}{(2n)!}$$

5. What function has Maclaurin series $x-\frac{x^3}{2!}+\frac{x^5}{4!}-\frac{x^7}{6!}+\frac{x^9}{8!}-\ldots$?
6. a) Use a third Taylor polynomial at $x=100$ to approximate $\sqrt{102}$. b) Give an upper bound for the error in using this approximation. (Hint 1)
7. Use a Maclaurin series to approximate $\int_0^1 e^{-x^3}\,dx$ with error less than $1/100$. Don't go too far. Leave your answer as fraction. (Decimals more sensible, but no calc. on exams).
8. Let v be the vector $3\mathbf{i}-4\mathbf{j}+5\mathbf{k}$. a) Find a non-0 vector v' orthogonal to v. (Hint 2) b) Find a vector orthogonal to v and v'.
9. Find a vector of length 3 pointing in the opposite direction from the vector $2\mathbf{i}-4\mathbf{j}+4\mathbf{k}$.
10. The vectors $2\mathbf{i}+3\mathbf{j}$, $4\mathbf{i}+\mathbf{j}$ and $5\mathbf{i}+y\mathbf{j}$ have their tails at the origin. Find the value of y that will make their heads collinear.
11. Find the projection of $-\mathbf{i}+5\mathbf{j}-2\mathbf{k}$ on $3\mathbf{i}-\mathbf{j}$.
12. Find the angle between the planes $2x+2y=7$ and $y+z=-1$.
13. a) Find the area of the triangle with vertices $(0,1,1)$, $(-1,2,4)$, $(3,0,1)$. b) Find the equation of the plane containing the triangle.
14. There is a line L parametrized by $x=1+2t$, $y=3+4t$, $z=5+6t$. The line passes through some point P whose z-coordinate is -7. Find equation for the plane through P normal to L.
15. a) Parametrize the intersection of $x^2+y^2+z^2=25$ and the plane $x=4$. b) Which axis is perpendicular to the plane $x=4$?

16. Consider the parametrization $x = 2 - t - t^3$ and $y = 3t^2 - 6t + 1$. Find dy/dx at the point with (x, y)-coordinates. $(0, -2)$.

17. Find parametric equations for the line through $(2, 1, 1)$ which is normal to the plane containing $(0, 1, 1)$, $(-1, 2, 4)$ and $(3, 0, 1)$.

18. Parametrize the line of intersection of the $6x - 3y - 5z = 11$ and $3x + 9y + z = 2$

19. Parametrize the intersection of the surfaces $4x^2 + z^2 = 16$ and $y = z$.

20. Find the equation of the hyperbola with asymptotes $y = 3x$ and $y = -3x$ passing through the point $(0, 2)$.

21. Two trees grow 100 feet apart. A 120 foot rope is attached to the first tree at a height of 100 feet and to the second tree at a height of 50 feet. A Malawian child folds an empty plastic water bottle over the rope and rappels from the high end of the rope to the low end of the rope. At each moment, the rope is taut. The point where the water bottle is folded over the rope traces a path. Now ignore the numbers and state what kind of curve the path makes: line, ellipse, parabola or hyperbola?

22. Graph $y - z = x^2$.

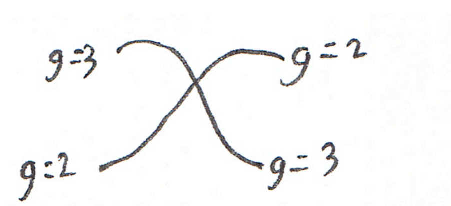
23. Graph $x^2 + y^2 = z^2 - 4$.

24. Describe the following regions and graphs a) $x^2 + y^2 + z^2 \leq 4$, $z \geq 0$ *b)* $x^2 + y^2 + z^2 \leq 4$, $z = 0$ c) $x^2 + y^2 + z^2 = 4$, $z \geq 0$ *d)* $x^2 + y^2 + z^2 = 4$, $z = 0$

25. Sketch the level surface $f(x, y, z) = z^2 - x^2 - y^2$ for $f(x, y, z) = 4$.

26. Sketch the domain of $f(x, y) = \sqrt{y - 3x}$.

27. In the picture below are level curves of $g(x, y) = 2$ and $g(x, y) = 3$ for some function $g(x, y)$. What's wrong?

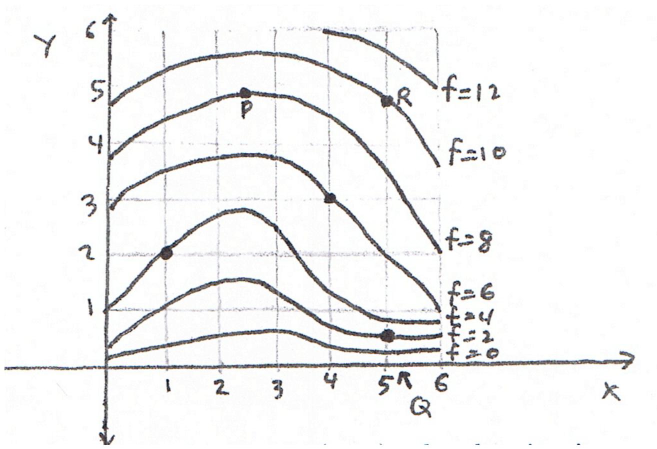


28. Let $f(x, y, z) = xy + 3y^2z^3 + \ln(xz)$. Compute $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial y \partial z}$.

29. Let $f(x, z) = x^2z + e^{xz+z^2}$ and $x = r\cos(\theta)$, $z = r\sin(\theta)$. Use the chain rule to compute $\partial f/\partial\theta$ at $(r, \theta) = (3, \pi/2)$.

30. In xyz-space, the temperature at the point (x, y, z) is given by $T(x, y, z) = 4xy + 3z^2$. You're at $(2, 0, 1)$ where the temperature is $3°$. Brrrr! (a) In what direction should you go, from $(2, 0, 1)$, in order to warm up the fastest? (Hint 3) (b) What is the directional derivative in that direction at the point $(2, 0, 1)$? (c) If you walk .2 units in that direction, by about how many degrees will you warm up?

31. (See Hint 4 for hints to all parts.) f is a nice function of x and y (i.e. f is continuous and its partials exist). In the diagram below are shown the level curves $f(x, y) = 0, 2, 4, 6, 8, 10, 12$.



(a) Let $u_1 = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$. Estimate $D_{u_1}f(4, 3)$.

(b) Draw on the diagram a unit vector pointing in the same direction as $\nabla f(1, 2)$.

(c) At P, draw a unit vector and label it u_2, such that $D_{u_2}f(P) < 0$.

(d) Estimate $\frac{\partial f}{\partial x}(P)$.

(e) Which appears bigger, $\frac{\partial f}{\partial y}(Q)$ or $\frac{\partial f}{\partial y}(R)$?

32. A sheet of metal of varying density occupies the xy-plane. At the point (x, y), the density is $100 + .1xy + .1y^2$. We move away from the point $(10, 10)$ in the direction $\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$. Use the direc'l derivative to estimate about how far we'll have to go for the density to increase by 6.

33. Find the equation of the tangent plane to the sphere $x^2 + y^2 + z^2 = 14$ at $(1, 2, 3)$.

34. The tangent plane to $z = f(x, y)$ at the point $(1, 2, 9)$ is $z = 3x + 5y - 4$. Find $\nabla f(1, 2)$. (Hint 5)

35. Find all local maxima, minima and saddlepoints for $f(x, y) = \frac{1}{3}x^3 - x + xy^2$.

36. Let $f(x, y) = 2 + \ln(1 + 3x - 4y)$. Use a linearization to approximate $f(.1, .2)$. (Hint 6)

37. a) Find the maximum and minimum values of $f(x, y) = -x^2 + 4x - y^2 - 4y$ on the circle $x^2 + y^2 = 9$, first using Lagrange multipliers, then again by parametrizing the circle. (Hint 5) b) Find the maximum and minimum values of $f(x, y)$ over the disk $x^2 + y^2 \leq 9$. This will require a calculator.

38. Find the minimum value of $x + y^2 + z^3$ along the part of the plane $x + y + z = 2$ in the first octant $(x, y, z \geq 0)$. The solution found by Lagrange multipliers gives the minimum (so don't parametrize the border). Use common sense for the maximum value.

Hints:

1. Let $f(x) = \sqrt{x}$ and find the 3rd Taylor polynomial at 100. Then plug in $x = 102$.
2. Find three scalars (to put in front of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$) so that the dot product works out to be 0.
3. $\nabla f(2, 0, 1)$.
4. a. If you move one unit in that direction, about how much does the function increase? b. Orthogonal to level curve in direction of (greatest) increase. c. Point in any direction of decrease (so toward $f = 8$ level curve, not $f = 12$). d. Walk in $x+$ direction for a bit, stay on level curve, so no change in function. e. At Q, only need to go up .25 units for function to go up 2 (estimated partial: $2/.25 = 8$). At R, need to go up 1 unit for function to go up 2 (estimated partial: $2/1 = 2$).
5. Tangent plane is $z = f_x(a, b)(x-a)+f_y(a, b)(x-b)+f(a, b)$. So coefficient of x is $f_x(1, 2) = 3$, coefficient of y is $f_y(1, 2) = 5$ and $\nabla f(1, 2) = f_x(1, 2)\mathbf{i} + f_y(1, 2)\mathbf{j}$.
6. Compute linearization at $(x, y) = (0, 0)$ since that's near $(.1, .2)$ and easy to work with.

Answers:

1. $-\frac{9}{x^2}\cosh(\frac{3}{x})$
2. $5/4$
3. e
4. $a.\ 1/10 \quad b.\ -2 \quad c.\ \cos(5)$
5. $x\cos(x)$
6. $a)\ 10 + \frac{1}{10} - \frac{1}{2000} + \frac{1}{200000}$ $b)\ 1/16000000$
7. $169/210$
8. $a)\ 4\mathbf{i} + 3\mathbf{j}$ $b)\ -3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ (there are many correct answers to a and b)
9. $-\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$
10. $y = 0$
11. $-\frac{12}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$
12. $\pi/3$

13. $a)$ $\sqrt{94}/2$ $b)$ $3x + 9y - 2z = 7$

14. $2(x + 3) + 4(y + 5) + 6(z + 7) = 0$.

15. a) $x = 4$, $y = 3\cos(t)$, $z = 3\sin(t)$, $0 \le t \le 2\pi$ b) x-axis.

16. 0

17. $x = 2 + 3t, y = 1 + 9t, z = 1 - 2t$

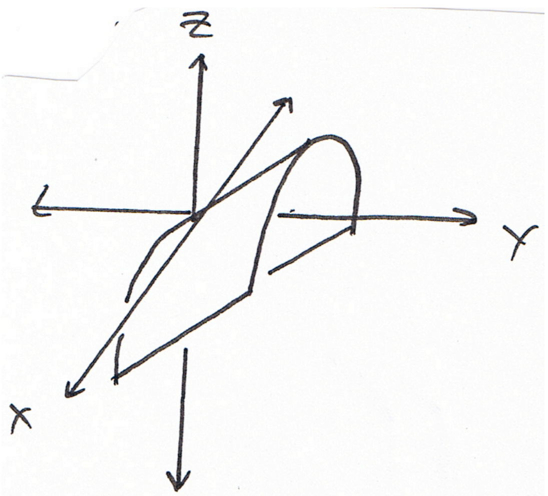
18. $x = 1 + 2t$, $y = -t$, $z = -1 + 3t$. (Line parallel to $2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$).

19. $x = 2\cos(t)$, $y = 4\sin(t)$, $z = 4\sin(t)$.

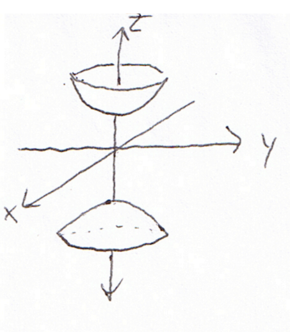
20. $y^2 - 9x^2 = 4$

21. Ellipse.

22. The graph is

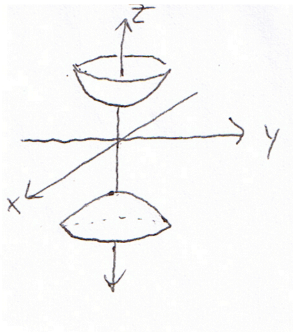


23. Hyperboloid of two sheets. It is symmetric about the z-axis (its central axis). The z-intercepts (its vertices) are $(0, 0, -2)$ and $(0, 0, 2)$.

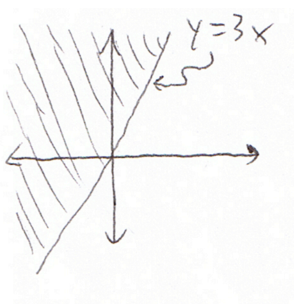


24. a) Part of the solid ball, $r = 2$, center=origin, that is above and in the xy-plane. b) Solid disk in xy-plane, $r = 2$, ctr=orig. c) upper half of sphere, $r = 2$, ctr=orig, above and in the xy-plane. d) circle, $r = 2$, ctr=orig, in xy-plane.

25. The graph looks like



26. The shaded region:



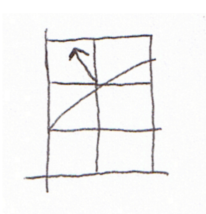
27. At the point P of intersection, $g(P) = 2$ and $g(P) = 3$; impossible.

28. $\frac{-1}{x^2}, 18yz^2$

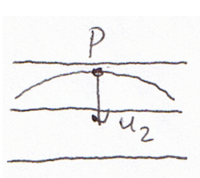
29. $(r, \theta) = (3, \pi/2)$, $(x, z) = (0, 3)$, $[2xz + ze^{xz+z^2}](-r\sin(\theta)) + [x^2 + (x + 2z)e^{xz+z^2}]r\cos(\theta)$ is $-9e^9$

30. *a*) $8\mathbf{j} + 6\mathbf{k}$ *b*) 10 *c*) 2

31. *a*) 2 *b*)



c)



d) 0 *e*) $\frac{\partial f}{\partial y}(Q)$

32. Directional derivative at $(10, 10)$ in that direction is 3 so about 2 units.

33. $2(x-1)+4(y-2)+6(z-3)=0$.

34. $3\mathbf{i}+5\mathbf{j}$.

35. saddle: $(0,\pm 1)$, local max:$(-1,0)$, local min:$(1,0)$

36. 1.5

37. $a) -9 \pm 24/\sqrt{2} \quad b) \ -9-24/\sqrt{2}, 8$

38. $\min: \ \frac{7}{4}-\frac{2}{3}\sqrt{1/3}, \quad \max: \ \ 8$